LOGARITHMS *A Basic Concept*

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01. **Definition :** A number 'x' is called the logarithm of a number 'N' to the base 'a' if $a^x = N$ and a > 0, a ≠ 1 and N > 0.

That is, if a > 0, a ≠ 1, N > 0 and $a^x = N$ then, $\log_a N = x$.

Case I : When a = 1.

Let $\log_1 N = x \Rightarrow 1^x = N$ [From definition

$\Rightarrow N = 1$.

So, if $\log_1 2$ is defined then, $2 = 1$ which is absurd. Hence for log to be defined, $a \neq 1$.

Case II : When a = 0.

Let $\log_0 N = x \Rightarrow 0^x = N$ [From definition

$\Rightarrow N = 0$.

So, if $\log_0 2$ is defined then, $2 = 0$ which is absurd. Hence for log to be defined, $a \neq 0$.

Similarly, it can be shown that if a is negative (a < 0) then the definition fails. Hence a > 0, for log to be defined.

Case III : When N < 0.

Let $\log_a N = x \Rightarrow a^x = N$ [From definition

$\Rightarrow (\text{Positive number})^x = \text{Negative number}$ [∵ a > 0, as explained in Case II.

$\Rightarrow$ Positive number = Negative number , which is wrong.

Hence, definition fails if N < 0.

Similarly, N = 0 will mean 'Positive number = 0'. So $N \neq 0$.

$\therefore$ $N > 0$ is necessary for log to be defined.

02. **System *Of* Logarithm :** Logarithm of numbers to the base 10 are known as '**common logarithm**'. Whenever log is used for numerical calculations, the base is generally taken as 10. In all such cases, if base is not given, it is assumed to be 10.

For theoretical investigations, another quantity **'e'** is used as base of log. Logarithms to the base e are called **natural logarithm or Napier logarithm**.

Abbreviations : $\log_{10} x = \log x$ or lg x and, $\log_e x = \ln x$.

03. **Some Important Results :**

A. $\log_a 1 = 0$.

Proof : Let $\log_a 1 = x \Rightarrow a^x = 1 = a^0 \Rightarrow x = 0$ $\therefore$ $\log_a 1 = 0$.

B. $\log_a a = 1$.

Proof : Let $\log_a a = x \Rightarrow a^x = a = a^1 \Rightarrow x = 1$ $\therefore$ $\log_a a = 1$.

C. $a^{\log_a N} = N$.

Proof : Let $\log_a N = x \Rightarrow a^x = N$ $\therefore$ $a^{\log_a N} = N$.

D. $\log_m (ab) = \log_m a + \log_m b$, (a > 0, b > 0)

Proof : Let $\log_m a = x$ and $\log_m b = y \Rightarrow m^x = a$ and $m^y = b$

$\therefore$ $ab = m^x m^y = m^{x+y}$ …(i)

Let $\log_m(ab) = z \Rightarrow m^z = ab \ldots(ii)$

By (i) & (ii), $m^z = m^{x+y} \Rightarrow x + y = z$

$\therefore \ \log_m(ab) = \log_m a + \log_m b$.

E. $\log_m\left(\frac{a}{b}\right) = \log_m a - \log_m b$, $(a > 0, b > 0)$

Proof : Let $\log_m a = x$ and $\log_m b = y \Rightarrow m^x = a$ and $m^y = b$

$\therefore \ \frac{a}{b} = \frac{m^x}{m^y} = m^{x-y} \ldots(i)$

Let $\log_m\left(\frac{a}{b}\right) = z \Rightarrow m^z = \frac{a}{b} \ldots(ii)$

By (i) & (ii), $m^z = m^{x-y} \Rightarrow x - y = z$

$\therefore \ \log_m\left(\frac{a}{b}\right) = \log_m a - \log_m b$.

F. $\log_m N^k = k \log_m N$.

Proof : Let $\log_m N = x \Rightarrow m^x = N \ldots(i)$

Also let $\log_m N^k = y \Rightarrow m^y = N^k = (m^x)^k$ [By (i)

$\Rightarrow m^y = m^{kx} \Rightarrow y = kx$ $\therefore \ \log_m N^k = k \log_m N$

G. (i) $\log_b a = \log_c a . \log_b c$

Proof : Let $\log_b a = x \Rightarrow b^x = a$, $\log_c a = y \Rightarrow c^y = a$ and $\log_b c = z \Rightarrow b^z = c$

So, $b^x = c^y \Rightarrow b^x = (b^z)^y = b^{yz} \Rightarrow x = yz$ Hence $\log_b a = \log_c a . \log_b c$.

(ii) $\log_b a = \frac{\log_c a}{\log_c b} = \frac{\log a}{\log b}$. (iii) $\log_b a = \frac{\log_a a}{\log_a b} = \frac{1}{\log_a b}$.

H. Base Power Formula : $\log_{(a^k)} N = \frac{1}{k}\log_a N$.

Proof : We have $\log_{(a^k)} N = \frac{\log N}{\log a^k} = \frac{\log N}{k \log a} = \frac{1}{k}.\frac{\log N}{\log a} = \frac{1}{k}\log_a N$.

Logarithm on napierian base can be converted in logarithm of common base (when base is 10).

That is, $\log_e x = \frac{\log_{10} x}{\log_{10} e} \Rightarrow \log_{10} x = \log_{10} e \, . \log_e x$ $\therefore \log_{10} x = 0.434 \log_e x$

Here $\frac{1}{\log_e 10} = \log_{10} e$ is called **'modulus'** of common system.

◘ Question Based on Logarithms

Q01. Obtain $f(\ln x)$, where $f(x) = \ln x$.

Q02. Is it true that $x = e^{\log x}$ for all real x?

Q03. Given that $f(x) = \sqrt{x \log_e x}$ then write the value of $f(e)$.

Simplify (Make the base of log as 'e' in all the cases below) :

Q04. $\log_5(\log x)$

Q05. $10^{5\log_{10} x}$

Q06. $\log_{x^7} x$

Q07. $\log_x x$

Q08. $\log_{\sqrt{x}} x$

Q09. $\log_x \sqrt{x}$.

Write the domain of following functions :

Q10. $f(x) = e^x \log |x|$

Q11. $y = \log |x|$

Q12. $f(x)=\log_{10}(x-3)$ **Q13.** $y=\dfrac{x}{\log_{10}(1-x)}$

Q14. $f(x)=\dfrac{\log x}{\sqrt{1-9x^2}}$.

Solutions for the Questions Based On Logarithms

Q01. Here $f(x)=ln\ x$ so, $f(ln\ x)=ln(ln\ x)$ or, $\log_e(\log_e x)$.

Q02. As log x is defined for all $x\in R^+$ so, $x=e^{\log x}$ is true only when $x\in R^+$.

Q03. Here $f(x)=\sqrt{x\log_e x}$ $\therefore f(e)=\sqrt{e\log_e e}=\sqrt{e\times 1}=\sqrt{e}$

Q04. $\log_5(\log x)=\dfrac{\log_e(\log x)}{\log_e 5}$

Q05. $10^{5\log_{10}x}=10^{\log_{10}x^5}=x^5$

Q06. $\log_{x^7}x=\dfrac{\log_e x}{\log_e x^7}=\dfrac{\log_e x}{7\log_e x}=\dfrac{1}{7}$

Q07. $\log_x x=\dfrac{\log_e x}{\log_e x}=1$

Q08. $\log_{\sqrt{x}}x=\dfrac{\log_e x}{\log_e\sqrt{x}}=\dfrac{\log_e x}{\frac{1}{2}\log_e x}=2$

Q09. $\log_x\sqrt{x}=\dfrac{\log_e\sqrt{x}}{\log_e x}=\dfrac{\frac{1}{2}\log_e x}{\log_e x}-\dfrac{1}{2}$.

Q10. Note that e^x is defined for all $x\in R$ and, log |x| is defined for all $x\in R-\{0\}$.

Hence domain of $f(x)=e^x\log|x|$ is $x\in R-0$.

Q11. As log |x| is defined for all $x\in R-\{0\}$ so, domain of $y=\log|x|$ is $x\in R-0$.

Q12. Domain of $f(x)=\log_{10}(x-3)$ is $x\in(3,\infty)$.

☞ Note that we must have $x-3>0$ for $\log_{10}(x-3)$ to be defined.

Q13. Here y shall be defined only when $\log_{10}(1-x)\neq 0$ *i.e.*, $1-x\neq 1 \Rightarrow x\neq 0$.

Also, $\log_{10}(1-x)$ is defined iff $1-x>0$ i.e., $x<1$.

Hence domain of y is $x\in(-\infty,1)-\{0\}$.

Q14. Do yourself. Ans. $x\in\left(0,\dfrac{1}{3}\right)$.