# CBSE 2019 DELHI ANNUAL EXAMINATION

**(Series BVM/1 Code No. 65/1/1 : Delhi Region)**

**Max. Marks : 100** **Time Allowed : 3 Hours**

## SECTION – A

**Q01.** If A and B are square matrices of the same order 3, such that $|A| = 2$ and $AB = 2I$, write the value of $|B|$.

**Sol.** As $AB = 2I$ $\Rightarrow |AB| = |2I|$ $\Rightarrow |A||B| = 2^3 |I|$ $\Rightarrow 2|B| = 8\times 1$ $\therefore |B| = 4$.

# Note that the order of A and B is 3 and AB = 2I so, I is also of order 3.

**Q02.** If $f(x) = x+1$, find $\frac{d}{dx}(fof)(x)$.

**Sol.** We have $(fof)(x) = f(f(x)) = f(x+1) = x+1+1 = x+2$ $\therefore \frac{d}{dx}(fof)(x) = \frac{d}{dx}(x+2) = 1$.

**Q03.** Find the order and degree of the differential equation $x^2\frac{d^2y}{dx^2} = \left\{1+\left(\frac{dy}{dx}\right)^2\right\}^4$.

**Sol.** Order is 2 and degree is 1.

**Q04.** If a line makes angles $90^\circ, 135^\circ, 45^\circ$ with the x, y and z axes respectively, find its direction cosines.

**OR** Find the vector equation of the line which passes through the point (3, 4, 5) and is parallel to the vector $2\hat{i}+2\hat{j}-3\hat{k}$.

**Sol.** Here $\alpha = 90^\circ, \beta = 135^\circ, \gamma = 45^\circ$.

Then direction cosines of the line are $\cos 90^\circ, \cos 135^\circ, \cos 45^\circ$ i.e., $0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$.

**OR** The vector equation of the line is $\vec{r} = 3\hat{i}+4\hat{j}+5\hat{k}+\lambda(2\hat{i}+2\hat{j}-3\hat{k})$.

# Note we have used $\vec{r} = \vec{a}+\lambda\vec{b}$.

## SECTION – B

**Q05.** Examine whether the operation * defined on R by $a*b = ab+1$ is (i) a binary or not? (ii) if a binary operation, is it associative or not?

**Sol.** Since $ab+1 \in R\ \forall\ a,b \in R$. Therefore $a*b \in R\ \forall\ a,b\in R$.

(i) Hence * is binary operation.

(ii) Let $a,b,c \in R$.

Now $a*(b*c) = a*(bc+1) = abc+a+1$ and $(a*b)*c = (ab+1)*c = abc+c+1$.

As $a*(b*c) \neq (a*b)*c$, so * isn't associative.

**Q06.** Find a matrix A such that $2A-3B+5C = O$, where $B = \begin{bmatrix}-2 & 2 & 0\\ 3 & 1 & 4\end{bmatrix}$ and $C = \begin{bmatrix}2 & 0 & -2\\ 7 & 1 & 6\end{bmatrix}$.

**Sol.** As $2A-3B+5C = O$ $\Rightarrow 2A = 3B-5C$ $\Rightarrow 2A = \begin{bmatrix}-6 & 6 & 0\\ 9 & 3 & 12\end{bmatrix} - \begin{bmatrix}10 & 0 & -10\\ 35 & 5 & 30\end{bmatrix}$

$\Rightarrow 2A = \begin{bmatrix}-16 & 6 & 10\\ -26 & -2 & -18\end{bmatrix}$ $\therefore A = \begin{bmatrix}-8 & 3 & 5\\ -13 & -1 & -9\end{bmatrix}$.

**Q07.** Find : $\int \frac{\sec^2 x dx}{\sqrt{\tan^2 x+4}}$.

**Sol.** Let $I = \int \frac{\sec^2 x dx}{\sqrt{\tan^2 x+4}}$ Put $\tan x = t \Rightarrow \sec^2 x dx = dt$

$\therefore I = \int \frac{dt}{\sqrt{t^2+2^2}}$ $\Rightarrow I = \log\left|t+\sqrt{t^2+4}\right| + C$ $\therefore I = \log\left|\tan x + \sqrt{\tan^2 x + 4}\right| + C$

**Q08.** Find : $\int \sqrt{1-\sin 2x}\,dx,\ \frac{\pi}{4} < x < \frac{\pi}{2}$. **OR** Find : $\int \sin^{-1}(2x)dx$.

**Sol.** $\int \sqrt{1-\sin 2x}\,dx = \int \sqrt{(\sin x - \cos x)^2}\,dx = \int (\sin x - \cos x)dx = -\cos x - \sin x + C$.

# Note that $\sin x - \cos x > 0$ if $\frac{\pi}{4} < x < \frac{\pi}{2}$.

**OR** Let $I = \int \sin^{-1}(2x)dx = \int \sin^{-1}(2x).\,1\,dx = \sin^{-1}(2x)\int 1\,dx - \int\left\{\frac{d}{dx}(\sin^{-1}(2x))\int 1\,dx\right\}$

$\Rightarrow I = x\sin^{-1}(2x) - \int \frac{2x}{\sqrt{1-4x^2}}dx$ $\Rightarrow I = x\sin^{-1}(2x) + \frac{1}{4}\int \frac{-8x}{\sqrt{1-4x^2}}dx$

$\Rightarrow I = x\sin^{-1}(2x) + \frac{1}{4}\times 2\sqrt{1-4x^2} + C$ $\therefore I = x\sin^{-1}(2x) + \frac{1}{2}\sqrt{1-4x^2} + C$.

# Note that $\int \frac{f'(x)}{\sqrt{f(x)}}dx = 2\sqrt{f(x)} + C$.

**Q09.** Form the differential equation representing the family of curves $y = e^{2x}(a+bx)$, where 'a' and 'b' are arbitrary constants.

**Sol.** We have $y = e^{2x}(a+bx)$ $\Rightarrow e^{-2x}y = a+bx$

$\Rightarrow e^{-2x}y' + ye^{-2x}(-2) = 0 + b\times 1$ $\Rightarrow e^{-2x}\{y'-2y\} = b$

$\Rightarrow e^{-2x}\{y''-2y'\} + e^{-2x}(-2)\{y'-2y\} = 0$ $\therefore y'' - 4y' + 4y = 0$.

**Q10.** If the sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

**OR** If $\vec{a} = 2\hat{i}+3\hat{j}+\hat{k},\ \vec{b} = \hat{i}-2\hat{j}+\hat{k}$ and $\vec{c} = -3\hat{i}+\hat{j}+2\hat{k}$, find $[\vec{a}\ \vec{b}\ \vec{c}]$.

**Sol.** Let $\vec{a}$ and $\vec{b}$ be two unit vectors. So, $|\vec{a}| = 1,\ |\vec{b}| = 1$ and $|\vec{a}+\vec{b}| = 1$.

As $|\vec{a}+\vec{b}| = 1$ $\Rightarrow |\vec{a}+\vec{b}|^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b}) = 1$ $\Rightarrow a^2 + 2\vec{a}.\vec{b} + b^2 = 1$

$\Rightarrow 1^2 + 2\vec{a}.\vec{b} + 1^2 = 1$ $\Rightarrow 2\vec{a}.\vec{b} = -1 \ldots$(i)

Now $|\vec{a}-\vec{b}|^2 = (\vec{a}-\vec{b}).(\vec{a}-\vec{b}) = a^2 - 2\vec{a}.\vec{b} + b^2 = 1^2 - (-1) + 1^2 = 3$ [By (i)

Therefore, $|\vec{a}-\vec{b}| = \sqrt{3}$.

**Alternatively,** $|\vec{a}-\vec{b}|^2 + |\vec{a}+\vec{b}|^2 = 2\left[|\vec{a}|^2 + |\vec{b}|^2\right]$ $\Rightarrow |\vec{a}-\vec{b}|^2 + 1^2 = 2\left[1^2+1^2\right]$

$\Rightarrow |\vec{a}-\vec{b}|^2 = 3$ $\therefore |\vec{a}-\vec{b}| = \sqrt{3}$.

**OR** $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -2 & 1 \\ -3 & 1 & 2 \end{vmatrix} = 2(-5) - 3(5) + 1(-5) = -30$.

**Q11.** A die marked 1, 2, 3 in red and 4, 5, 6 in green is tossed. Let A be the event "number is even" and B be the event "number is marked with red".
Find whether the events A and B are independent or not.

**Sol.** Here A = {2, 4, 6} and B = {1, 2, 3} and $A \cap B = \{2\}$

Now $P(A) = \frac{3}{6} = \frac{1}{2},\ P(B) = \frac{3}{6} = \frac{1}{2}$ and $P(A\cap B) = \frac{1}{6} \neq P(A)P(B)$.

Hence A and B are not independent.

**Q12.** A die is thrown 6 times. If "getting an odd number" is a "success", what is the probability of (i) 5 successes? (ii) at most 5 successes?

**OR** The random variable X has a probability distribution P(X) of the following form, where 'k' is some number.

$$P(X=x)=\begin{cases} k, \text{ if } x=0 \\ 2k, \text{ if } x=1 \\ 3k, \text{ if } x=2 \\ 0, \text{ otherwise} \end{cases}.$$

Determine the value of k.

**Sol.** Here n = 6, p : probability of getting an odd number on the die. So, $p=3/6=1/2,\ q=1/2$.

As $P(X=r)={}^6C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{6-r}={}^6C_r\times\frac{1}{2^6}$

(i) $P(X=5)={}^6C_5\times\frac{1}{2^6}=\frac{6}{64}$

(ii) $P(X\le 5)=1-P(X>5)=1-P(X=6)=1-{}^6C_6\times\frac{1}{2^6}=1-\frac{1}{64}=\frac{63}{64}$.

**OR** Since $\sum P(X)=1$ $\quad\therefore P(X=0)+P(X=1)+P(X=2)+P(X=3)+...=1$

$\Rightarrow k+2k+3k+0+...=1 \qquad \therefore k=1/6$.

## SECTION – C

**Q13.** Show that the relation R on $\mathbb{R}$ defined as $R=\{(a,b):a\le b\}$, is reflexive, and transitive but not symmetric.

**OR** Prove that the function $f:N\to N$, defined by $f(x)=x^2+x+1$ is one-one but not onto. Find inverse of $f:N\to S$, where S is range of f.

**Sol.** Here (a, a) $\in R\ \forall a\in\mathbb{R}$ as $a\le a$ is true. So, R is reflexive.

Let $a,b,c\in\mathbb{R}$. Let $(a,b)\in R$ and $(b,c)\in R$.

That is, $a\le b$ and $b\le c$, which clearly, implies $a\le c$. Hence, $(a,c)\in R$. So, R is transitive.

Now let $a=1, b=2$.

We can notice that $(1,2)\in R$ as $1\le 2$ is true but, $(2,1)\notin R$ as $2\le 1$ is false.

So, R isn't symmetric, as $(a,b)\in R$ does not imply $(b,a)\in R$.

**OR** Here $f:N\to N$, defined by $f(x)=x^2+x+1$.

Let $x_1, x_2\in N$ and $f(x_1)=f(x_2)$. That is, $x_1^2+x_1+1=x_2^2+x_2+1 \Rightarrow x_1^2-x_2^2=x_2-x_1$

$\Rightarrow (x_1-x_2)(x_1+x_2)+(x_1-x_2)=0 \qquad \Rightarrow (x_1-x_2)(x_1+x_2+1)=0$

$\Rightarrow (x_1-x_2)=0 \qquad \therefore x_1=x_2 \qquad [\because x_1+x_2+1\ne 0\ \forall x_1,x_2\in N$

So, f is one-one.

Let $y=f(x)$ and $y\in N$.

So, $y=x^2+x+1 \qquad \Rightarrow y=x^2+x+\frac{1}{4}+\frac{3}{4} \quad \Rightarrow y=\left(x+\frac{1}{2}\right)^2+\frac{3}{4} \quad \Rightarrow y-\frac{3}{4}=\left(x+\frac{1}{2}\right)^2$

$\Rightarrow 4y-3=(2x+1)^2 \quad \Rightarrow \pm\sqrt{4y-3}=(2x+1)$

$\Rightarrow x=\frac{-1\pm\sqrt{4y-3}}{2}\notin N$ for all $y\in N$. Hence f is not onto.

Also, if $f:N\to S$, where S is the range of f then, $x=\frac{-1+\sqrt{4y-3}}{2}\in N$ if $y\in S$.

Hence, $f^{-1}(y)=\frac{-1+\sqrt{4y-3}}{2}$.

# Note that if we take $x=\dfrac{-1-\sqrt{4y-3}}{2}$ then, $2x+1\neq-\sqrt{4y-3}$, as $2x+1\in N\ \forall x\in N$.

**Q14.** Solve : $\tan^{-1}4x+\tan^{-1}6x=\dfrac{\pi}{4}$.

**Sol.** $\tan^{-1}4x+\tan^{-1}6x=\dfrac{\pi}{4}$ $\Rightarrow\tan^{-1}\dfrac{4x+6x}{1-4x\times6x}=\dfrac{\pi}{4}$ $\Rightarrow\dfrac{10x}{1-24x^2}=\tan\dfrac{\pi}{4}$

$\Rightarrow 10x=1-24x^2$ $\Rightarrow 24x^2+10x-1=0$ $\Rightarrow(2x+1)(12x-1)=0$

$\therefore(2x+1)=0$, or $(12x-1)=0$ $\Rightarrow x=-1/2$, or $x=1/12$.

As $x=-1/2$ doesn't satisfy the equation. Hence, $x=1/12$ is the only solution.

**Q15.** Using properties of determinants, prove that $\begin{vmatrix}a^2+2a & 2a+1 & 1\\ 2a+1 & a+2 & 1\\ 3 & 3 & 1\end{vmatrix}=(a-1)^3$.

**Sol.** LHS : Let $\Delta=\begin{vmatrix}a^2+2a & 2a+1 & 1\\ 2a+1 & a+2 & 1\\ 3 & 3 & 1\end{vmatrix}$ By $C_1\to C_1-C_2$, $C_2\to C_2-3C_3$

$\Rightarrow\Delta=\begin{vmatrix}a^2-1 & 2a-2 & 1\\ a-1 & a-1 & 1\\ 0 & 0 & 1\end{vmatrix}$ Taking $(a-1)$ common from $C_1$ and $C_2$ both

$\Rightarrow\Delta=(a-1)^2\begin{vmatrix}a+1 & 2 & 1\\ 1 & 1 & 1\\ 0 & 0 & 1\end{vmatrix}$ Expanding along $R_3$

$\Rightarrow\Delta=(a-1)^2\{a+1-2\}=(a-1)^3=\text{RHS}$.

**Q16.** If $\log(x^2+y^2)=2\tan^{-1}\left(\dfrac{y}{x}\right)$, show that $\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$.

**OR** If $x^y-y^x=a^b$, find $\dfrac{dy}{dx}$.

**Sol.** We have $\log(x^2+y^2)=2\tan^{-1}\left(\dfrac{y}{x}\right)$ $\Rightarrow\dfrac{1}{x^2+y^2}\times\left\{2x+2y\dfrac{dy}{dx}\right\}=2\times\dfrac{1}{1+\dfrac{y^2}{x^2}}\times\left(\dfrac{xy'-y\times1}{x^2}\right)$

$\Rightarrow 2\times\dfrac{x+yy'}{x^2+y^2}=2\times\dfrac{xy'-y}{x^2+y^2}$ $\Rightarrow x+yy'=xy'-y$ $\Rightarrow x+y=xy'-yy'$ $\therefore\dfrac{dy}{dx}=\dfrac{x+y}{x-y}$

**OR** We've $x^y-y^x=a^b$ $\Rightarrow e^{\log x^y}-e^{\log y^x}=a^b$ $\Rightarrow e^{y\log x}-e^{x\log y}=a^b$

$\Rightarrow e^{y\log x}\times\left\{y\times\dfrac{1}{x}+\log x\times\dfrac{dy}{dx}\right\}-e^{x\log y}\left\{x\times\dfrac{1}{y}\times\dfrac{dy}{dx}+\log y\times1\right\}=0$

$\Rightarrow x^y\left\{\dfrac{y}{x}+\log x\times\dfrac{dy}{dx}\right\}-y^x\left\{\dfrac{x}{y}\times\dfrac{dy}{dx}+\log y\right\}=0$

$\Rightarrow\left\{x^{y-1}y+x^y\log x\times\dfrac{dy}{dx}\right\}-\left\{y^{x-1}x\times\dfrac{dy}{dx}+y^x\log y\right\}=0$

$\Rightarrow x^{y-1}y+\left[x^y\log x-y^{x-1}x\right]\times\dfrac{dy}{dx}-y^x\log y=0$ $\therefore\dfrac{dy}{dx}=\dfrac{y^x\log y-x^{y-1}y}{x^y\log x-y^{x-1}x}$.

**Q17.** If $y=(\sin^{-1}x)^2$, prove that $(1-x^2)\dfrac{d^2y}{dx^2}-x\dfrac{dy}{dx}-2=0$.

**Sol.** Here $y=(\sin^{-1}x)^2 \quad \Rightarrow y'=2(\sin^{-1}x)\times\frac{1}{\sqrt{1-x^2}} \quad \Rightarrow y'\sqrt{1-x^2}=2(\sin^{-1}x)$

$\Rightarrow y''\sqrt{1-x^2}+y'\times\frac{-2x}{2\sqrt{1-x^2}}=\frac{2}{\sqrt{1-x^2}} \quad \Rightarrow (1-x^2)y''-xy'=2$

$\therefore (1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}-2=0$.

**Q18.** Find the equation of tangent to the curve $y=\sqrt{3x-2}$ which is parallel to the line $4x-2y+5=0$. Also, write the equation of normal to the curve at the point of contact.

**Sol.** Slope of the given line $4x-2y+5=0$ is $-\frac{4}{-2}=2$.

Let $P(\alpha,\beta)$ be the point of contact on the curve $y=\sqrt{3x-2}$. So, $\beta=\sqrt{3\alpha-2}$...(i)

Now $\frac{dy}{dx}=\frac{3}{2\sqrt{3x-2}} \quad \Rightarrow \left.\frac{dy}{dx}\right]_{\text{at P}}=\frac{3}{2\sqrt{3\alpha-2}}=2$ (As tangent is parallel to the given line.

$\Rightarrow \frac{3}{4}=\sqrt{3\alpha-2} \quad \Rightarrow \frac{3}{4}=\beta$ (By (i)

So, $\frac{3}{4}=\sqrt{3\alpha-2} \quad \Rightarrow \frac{9}{16}+2=3\alpha \quad \Rightarrow \alpha=\frac{41}{48} \quad \therefore P\left(\frac{41}{48},\frac{3}{4}\right)$

Eq. of tangent : $y-\frac{3}{4}=2\left(x-\frac{41}{48}\right) \quad \Rightarrow 48y-36=96x-82 \quad \therefore 48x-24y=23$

Eq. of normal : $y-\frac{3}{4}=-\frac{1}{2}\left(x-\frac{41}{48}\right) \quad \Rightarrow 96y-72=-(48x-41) \quad \therefore 48x+96y=113$.

**Q19.** Find : $\int\frac{3x+5}{x^2+3x-18}dx$.

**Sol.** $\int\frac{3x+5}{x^2+3x-18}dx=\frac{3}{2}\int\frac{(2x+3)+\frac{1}{3}}{x^2+3x-18}dx=\frac{3}{2}\int\frac{2x+3}{x^2+3x-18}dx+\frac{1}{2}\int\frac{1}{x^2+3x-18}dx$

$\Rightarrow \quad =\frac{3}{2}\log\left|x^2+3x-18\right|+\frac{1}{2}\int\frac{1}{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}dx \quad \left[\because \int\frac{f'(x)}{f(x)}dx=\log|f(x)|+C\right.$

$\Rightarrow \quad =\frac{3}{2}\log\left|x^2+3x-18\right|+\frac{1}{2}\times\frac{1}{2(9/2)}\log\left|\frac{x+\frac{3}{2}-\frac{9}{2}}{x+\frac{3}{2}+\frac{9}{2}}\right|+C$

$\Rightarrow \quad =\frac{3}{2}\log\left|x^2+3x-18\right|+\frac{1}{18}\log\left|\frac{x-3}{x+6}\right|+C$.

**Q20.** Prove that $\int_0^a f(x)\,dx=\int_0^a f(a-x)dx$, hence evaluate $\int_0^{\pi}\frac{x\sin x}{1+\cos^2 x}dx$.

**Sol.** Consider $\int_0^a f(x)\,dx$.

Let $x=a-t \Rightarrow dx=-dt$. Also when $x=0 \Rightarrow t=a$ and, when $x=a \Rightarrow t=0$.

So, $\int_0^a f(x)\,dx=\int_a^0 f(a-t)(-dt)=-\left[-\int_0^a f(a-t)dt\right]=\int_0^a f(a-t)dt$

Hence, $\int_0^a f(x)\,dx = \int_0^a f(a-x)dx$. [Replacing t by x

Let $I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx$...(i)

$\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$ $\Rightarrow I = \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$...(ii)

By (i) and (ii), we get : $2I = \int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}dx$

Put $\cos x = t \Rightarrow \sin x dx = -dt$. When $x = 0 \Rightarrow t = 1$, when $x = \pi \Rightarrow t = -1$.

$\therefore 2I = -\pi \int_1^{-1} \frac{dt}{1+t^2} = \pi \int_{-1}^{1} \frac{dt}{1+t^2} = \pi\left[\tan^{-1} t\right]_{-1}^{1} = \pi\left[\tan^{-1} 1 - \tan^{-1}(-1)\right] = \pi\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$

$\Rightarrow I = \frac{\pi^2}{4}$ or, $\left(\frac{\pi}{2}\right)^2$.

**Q21.** Solve the differential equation : $x dy - y dx = \sqrt{x^2+y^2}dx$, given that $y = 0$ when $x = 1$.

**OR** Solve the differential equation : $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$, subject to the initial condition $y(0) = 0$.

**Sol.** We have $x dy - y dx = \sqrt{x^2+y^2}dx$ $\Rightarrow \frac{dy}{dx} - \frac{y}{x} = \sqrt{1+\frac{y^2}{x^2}}$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ $\Rightarrow v + x\frac{dv}{dx} - v = \sqrt{1+v^2}$ $\Rightarrow \int \frac{dv}{\sqrt{1+v^2}} = \int \frac{dx}{x}$

$\Rightarrow \log\left|v + \sqrt{1+v^2}\right| = \log|x| + \log|C|$ $\Rightarrow \log\left|\frac{y+\sqrt{x^2+y^2}}{x}\right| = \log|Cx|$

$\Rightarrow y + \sqrt{x^2+y^2} = kx^2$, where $k = \pm C$

It is given that $y = 0$ when $x = 1$ so, $0 + \sqrt{1^2+0^2} = k \times 1^2$ $\Rightarrow k = 1$.

Hence required solution is $y + \sqrt{x^2+y^2} = x^2$.

**OR** We have $(1+x^2)\frac{dy}{dx} + 2xy - 4x^2 = 0$ $\Rightarrow \frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$

On comparing with $\frac{dy}{dx} + P(x)y = Q(x)$, we get $P(x) = \frac{2x}{1+x^2}$, $Q(x) = \frac{4x^2}{1+x^2}$

Now I.F. $= e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1+x^2$

The solution is given by $y(1+x^2) = \int (1+x^2) \times \frac{4x^2}{1+x^2}dx + C$ $\Rightarrow y(1+x^2) = \frac{4}{3}x^3 + C$

Since $y(0) = 0$ so, $0(1+0^2) = \frac{4}{3} \times 0^3 + C \Rightarrow C = 0$.

Hence the required solution is $y = \frac{4}{3} \times \frac{x^3}{1+x^2}$.

**Q22.** If $\hat{i}+\hat{j}+\hat{k}$, $2\hat{i}+5\hat{j}$, $3\hat{i}+2\hat{j}-3\hat{k}$ and $\hat{i}-6\hat{j}-\hat{k}$ respectively, are the position vectors of points A, B, C and D, then find the angle between the straight lines AB and CD. Find whether $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear or not.

**Sol.** Given $\overrightarrow{OA}=\hat{i}+\hat{j}+\hat{k},\ \overrightarrow{OB}=2\hat{i}+5\hat{j},\ \overrightarrow{OC}=3\hat{i}+2\hat{j}-3\hat{k}$ and $\overrightarrow{OD}=\hat{i}-6\hat{j}-\hat{k}$.

The vectors parallel to the lines AB and CD are respectively, given by $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Also the angle between lines AB and CD will be the same as the angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$.

Now $\overrightarrow{AB}=\hat{i}+4\hat{j}-\hat{k},\ \overrightarrow{CD}=-2\hat{i}-8\hat{j}+2\hat{k}$.

Consider angle between $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be $\theta$ $\therefore \cos\theta=\dfrac{\overrightarrow{AB}.\overrightarrow{CD}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{CD}\right|}=\dfrac{(\hat{i}+4\hat{j}-\hat{k}).(-2\hat{i}-8\hat{j}+2\hat{k})}{\sqrt{1+16+1}\sqrt{4+64+4}}$

$$\Rightarrow \cos\theta=\frac{(\hat{i}+4\hat{j}-\hat{k}).(-2\hat{i}-8\hat{j}+2\hat{k})}{\sqrt{1+16+1}\sqrt{4+64+4}}=\frac{-2-32-2}{\sqrt{18}\sqrt{72}}=-1 \qquad \Rightarrow \theta=\pi \qquad [\because 0\le\theta\le\pi$$

So, the required angle between lines AB and CD is $\pi$ i.e., $180^\circ$.

Hence, the vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are collinear.

**Q23.** Find the value of $\lambda$, so that the lines $\dfrac{1-x}{3}=\dfrac{7y-14}{\lambda}=\dfrac{z-3}{2}$ and $\dfrac{7-7x}{3\lambda}=\dfrac{y-5}{1}=\dfrac{6-z}{5}$ are at right angle. Also, find whether the lines are intersecting or not.

**Sol.** Re-writing the lines in symmetrical form :

$$L_1:\frac{x-1}{-3}=\frac{y-2}{\frac{\lambda}{7}}=\frac{z-3}{2} \text{ and } L_2:\frac{x-1}{-\frac{3\lambda}{7}}=\frac{y-5}{1}=\frac{z-6}{-5}$$

The d.r.'s of lines are respectively $-3,\ \dfrac{\lambda}{7},\ 2;\ -\dfrac{3\lambda}{7},\ 1,\ -5$.

As the lines $L_1$ and $L_2$ are at right angle so, $(-3).\left(-\dfrac{3\lambda}{7}\right)+\left(\dfrac{\lambda}{7}\right).1+2.(-5)=0$

$$\Rightarrow \frac{9\lambda}{7}+\frac{\lambda}{7}=10 \qquad \Rightarrow \frac{10\lambda}{7}=10 \qquad \therefore \lambda=7.$$

$$\text{Since }\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}=\begin{vmatrix} 1-1 & 5-2 & 6-3 \\ -3 & 1 & 2 \\ -3 & 1 & -5 \end{vmatrix}=\begin{vmatrix} 0 & 3 & 3 \\ -3 & 1 & 2 \\ -3 & 1 & -5 \end{vmatrix}=-63\ne 0$$

Hence, the lines are non-intersecting.

## SECTION – D

**Q24.** If $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$, find $A^{-1}$. Hence, solve the following system of equations :

$x+y+z=6,\ x+2z=7,\ 3x+y+z=12$.

**OR** Find the inverse of the following matrix using elementary operations,

$$A=\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}.$$

**Sol.** Here $A=\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}$ $\Rightarrow |A|=\begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{vmatrix}=1(-2)-1(-5)+(1)=4\ne 0$ $\therefore A^{-1}$ exists.

Consider $A_{ij}$ be the cofactor of $a_{ij}$.

$$A_{11}=-2,\ A_{12}=5,\ A_{13}=1,$$
$$A_{21}=0,\ A_{22}=-2,\ A_{23}=2, \qquad \Rightarrow \text{adj.}A=\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$$
$$A_{31}=2,\ A_{32}=-1,\ A_{33}=-1$$

$\therefore A^{-1} = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$ $\left\{\because A^{-1} = \frac{1}{|A|}\text{adj.}A\right.$.

Consider the given systems of equations : $x+y+z=6,\ x+2z=7,\ 3x+y+z=12$

These equations can be expressed as : $AX=B$ where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix}$

Therefore, $X = A^{-1}B$ $\therefore X = \frac{1}{4}\begin{bmatrix} -2 & 0 & 2 \\ 5 & -2 & -1 \\ 1 & 2 & -1 \end{bmatrix}\begin{bmatrix} 6 \\ 7 \\ 12 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$

$\therefore$ by equality of matrices : $x=3,\ y=1,\ z=2$.

**OR** We have $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

Using $A = AI$, we get $\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 \to R_2 + R_1$, $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 \to R_2 + 2R_3$, $\begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 2R_2$ $\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} -1 & -2 & -4 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

By $R_1 \to R_1 + 2R_3$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$

As $I = AA^{-1}$ $\therefore A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$.

**Q25.** A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 $m^3$. If building of tank costs ₹70 per square metre for the base and ₹45 per square metre for the sides, what is the cost of least expensive tank?

**Sol.** Given volume of the box of height (depth) 2 m is, $Lbh = 8 \Rightarrow Lb \times 2 = 8 \Rightarrow b = \frac{4}{L}$

The cost of tank, $C = 70(L\times b) + 45(2\times L\times h) + 45(2\times b\times h) = 280 + 180(L+b)$

$\Rightarrow C = 280 + 180\left(L + \frac{4}{L}\right)$ $\therefore \frac{dC}{dL} = 0 + 180\left(1 - \frac{4}{L^2}\right)$ and, $\frac{d^2C}{dL^2} = 180\left(\frac{8}{L^3}\right)$

For local points of maxima and/or minima, we have $\frac{dC}{dL}=180\left(1-\frac{4}{L^2}\right)=0 \Rightarrow L=2m$

$\because \left.\frac{d^2C}{dL^2}\right]_{at\ L=2m}=180>0$

$\therefore$ C is least at L = 2m.

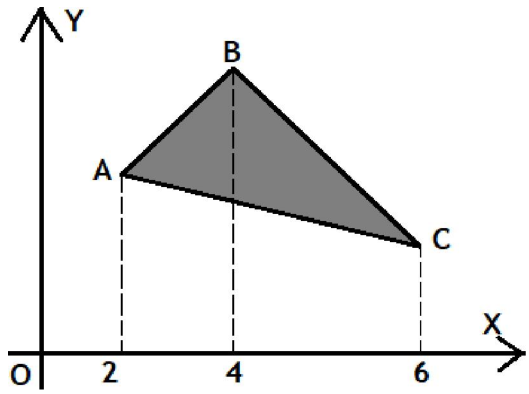
Now the least cost, $C=280+180\left(2+\frac{4}{2}\right)=280+180\times4=1000$.

Hence the least cost is ₹1000.

**Q26.** Using integration, find the area of the triangle ABC, whose vertices are A(2, 5), B(4, 7) and C(6, 2).

**OR** Find the area of the region lying about x-axis and included between the circle $x^2+y^2=8x$ and inside of the parabola $y^2=4x$.

**Sol.** We have A(2, 5), B(4, 7) and C(6, 2).




Eq. of AB : $y=x+3$,

Eq. of BC : $y=\frac{34-5x}{2}$,

Eq. of CA : $y=\frac{26-3x}{4}$.

Required area is given as,

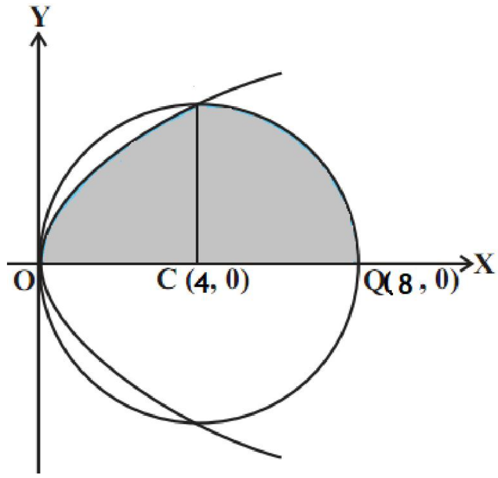
$$\Delta=\int_2^4(x+3)dx+\int_4^6\frac{34-5x}{2}dx-\int_2^6\frac{26-3x}{4}dx$$

$$\Rightarrow \Delta=\frac{1}{2}\left[(x+3)^2\right]_2^4-\frac{1}{20}\left[(34-5x)^2\right]_4^6+\frac{1}{24}\left[(26-3x)^2\right]_2^6$$

$$\Rightarrow \Delta=\frac{1}{2}[49-25]-\frac{1}{20}[16-196]+\frac{1}{24}[64-400]$$

$$\Rightarrow \Delta=12+9-14=7 \text{ sq. units}.$$

**OR** We have $x^2+y^2=8x$ …(i) and, $y^2=4x$ …(ii)




Solving (i) and (ii), we get : $x^2+4x=8x$

$\Rightarrow x^2-4x=0 \quad \Rightarrow x(x-4)=0 \quad \Rightarrow x=0,4$.

Also, $x^2-8x+y^2=0 \Rightarrow x^2-8x+16+y^2=16$

$\Rightarrow (x-4)^2+(y-0)^2=4^2 \qquad \therefore C(4,0),\ r=4$.

$$\text{Req. area}=2\left[\int_0^4 2\sqrt{x}dx+\int_4^8\sqrt{4^2-(x-4)^2}dx\right]$$

$$=\frac{8}{3}\left[x^{3/2}\right]_0^4+2\left[\frac{x-4}{2}\sqrt{16-(x-4)^2}+\frac{16}{2}\sin^{-1}\frac{x-4}{4}\right]_4^8$$

$$=\frac{8}{3}[8-0]+2\left[0+8\times\frac{\pi}{2}\right]-2[0+8\times0]$$

$$=\frac{64}{3}+8\pi \text{ sq. units}.$$

**Q27.** Find the vector and Cartesian equations of the plane passing through the points (2, 2, –1), (3, 4, 2) and (7, 0, 6). Also find the vector equation of a plane passing through (4, 3, 1) and parallel to the plane obtained above.

**OR** Find the vector equation of the plane that contains the lines $\vec{r}=(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})$ and the point (–1, 3, –4). Also, find the length of the perpendicular drawn from the point (2, 1, 4) to the plane, thus obtained.

**Sol.** Eq. of plane passing through the points (2, 2, –1), (3, 4, 2) and (7, 0, 6) is :

$$\begin{vmatrix} x-2 & y-2 & z+1 \\ 3-2 & 4-2 & 2+1 \\ 7-2 & 0-2 & 6+1 \end{vmatrix} = 0 \quad \Rightarrow \begin{vmatrix} x-2 & y-2 & z+1 \\ 1 & 2 & 3 \\ 5 & -2 & 7 \end{vmatrix} = 0$$

$\Rightarrow 20(x-2)+8(y-2)-12(z+1)=0 \ \Rightarrow 5x+2y-3z=17 \ldots$(i)

And, vector equation is $\vec{r}.(5\hat{i}+2\hat{j}-3\hat{k})=17$.

The d.r.'s of the normal to the plane (i) are 5, 2, –3.

As the d.r.'s of the plane through (4, 3, 1) and parallel to (i) will be proportional to the d.r.'s of the plane (i).

So, the eq. of this plane is $5(x-4)+2(y-3)-3(z-1)=0 \quad \Rightarrow 5x+2y-3z=23$

Therefore, the required vector eq. is $\vec{r}.(5\hat{i}+2\hat{j}-3\hat{k})=23$.

**OR** As the required plane $\pi$ (say) contains the line $\vec{r}=(\hat{i}+\hat{j})+\lambda(\hat{i}+2\hat{j}-\hat{k})$ so, the point (1, 1, 0) on the line will also lie on this plane. Also the plane $\pi$ contains (–1, 3, –4).

Let A(1, 1, 0) and B(–1, 3, –4). So, $\overrightarrow{AB}=-2\hat{i}+2\hat{j}-4\hat{k}$.

Normal vector to the plane $\pi$ can be obtained by $(\hat{i}+2\hat{j}-\hat{k})\times\overrightarrow{AB}$.

$$\therefore \text{Normal vector} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & -1 \\ -2 & 2 & -4 \end{vmatrix} = -6\hat{i}+6\hat{j}+6\hat{k}.$$

Therefore, the plane $\pi$ is : $-6(x-1)+6(y-1)+6(z-0)=0$ i.e., $x-y-z=0 \ldots$(i)

Now the length of the perpendicular drawn from the point (2, 1, 4) to the plane (i) is,

$$p=\frac{|1\times2-1\times1-1\times4+0|}{\sqrt{1^2+(-1)^2+(-1)^2}}=\frac{3}{\sqrt{3}}=\sqrt{3}\text{ units}.$$

**Q28.** A manufacturer has three machines operators A, B and C. The first operator A produces 1% of defective items, whereas the other two operators B and C produce 5% and 7% defective items respectively. A is on the job for 50% of the time, B on the job 30% of the time and C on the job for 20% of the time. All the items are put into one stockpile and then one item is chosen at random from this and is found to be defective. What is the probability that it was produced by A?

**Sol.** Let E : the chosen item is defective.

Also let $E_1$, $E_2$, $E_3$ be the events that the item was produced by operators A, B, C respectively.

We have $P(E_1)=50\%,\ P(E_2)=30\%,\ P(E_3)=20\%$,

$P(E|E_1)=1\%,\ P(E|E_2)=5\%,\ P(E|E_3)=7\%$.

Using Bayes' Theorem, $P(E_1|E)=\dfrac{P(E|E_1)P(E_1)}{P(E|E_1)P(E_1)+P(E|E_2)P(E_2)+P(E|E_3)P(E_3)}$

$$\Rightarrow P(E_1|E)=\frac{\frac{1}{100}\times\frac{50}{100}}{\frac{1}{100}\times\frac{50}{100}+\frac{5}{100}\times\frac{30}{100}+\frac{7}{100}\times\frac{20}{100}}$$

$$\therefore P(E_1|E)=\frac{5}{5+15+14}=\frac{5}{34}.$$

**Q29.** A manufacturer has employed 5 skilled men and 10 semi-skilled men and makes two models A and B of an article. The making of one item of model A requires 2 hours work by a skilled man and 2 hours work by a semi-skilled man. One item of model B requires 1 hour by a skilled man and 3 hours by a semi-skilled man. No man is expected to work more than 8 hours per day. The manufacturer's profit on an item if model A is ₹15 and on an item of model B is ₹10. How many of items of each model should be made per day in order to maximize daily profit? Formulate the above LPP and solve it graphically and find the maximum profit.

**Sol.** Let number of items of model A and B made per day be x and y, respectively.

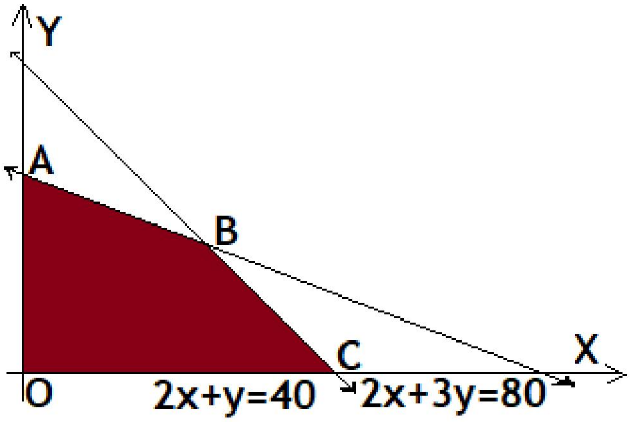
It's given that no man is expected to work more than 8 hours per day. Also 5 skilled men and 10 semi-skilled men are available so, at most 40 hours and 80 hours respectively time per day is available.

To maximize : $Z = 15x + 10y$ (in ₹).

Subject to constraints : $x \geq 0,\ y \geq 0,$

$2x + y \leq 40,\ 2x + 3y \leq 80$

| **Corner Points** | **Value of Z** |
|---|---|
| A(0, 80/3) | 800/3 |
| B(10, 20) | $350 \leftarrow$ max. |
| C(20, 0) | 300 |
| O(0, 0) | 0 |




Hence, maximum profit of ₹350 is obtained when 10 items of model A and 20 items of model B are made per day.

### (Series BVM/1 Code No. 65/1/2 : Delhi Region)

**Note** that only those questions from set 2 have been given which weren't in set 1.

**Q02.** If $f(x) = x + 7$ and $g(x) = x - 7,\ x \in R$ then, find $\dfrac{d}{dx}(fog)(x)$.

**Sol.** $(fog)(x) = f(g(x)) = f(x-7) = x - 7 + 7 = x$ $\quad \therefore \dfrac{d}{dx}(fog)(x) = \dfrac{d}{dx}(x) = 1$.

**Q03.** Find the value of $x - y$, if $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$.

**Sol.** $2\begin{bmatrix} 1 & 3 \\ 0 & x \end{bmatrix} + \begin{bmatrix} y & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix} \Rightarrow \begin{bmatrix} 2+y & 6 \\ 1 & 2x+2 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 8 \end{bmatrix}$

By equality of matrices, we get : $2 + y = 5,\ 2x + 2 = 8 \Rightarrow y = 3, x = 3$ $\quad \therefore x - y = 0$.

**Q06.** If $A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$, then find $A^2 - 5A$.

**Sol.** $A^2 = A.A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix}$

$\therefore A^2 - 5A = \begin{bmatrix} 5 & -1 & 2 \\ 9 & -2 & 5 \\ 0 & -1 & -2 \end{bmatrix} - \begin{bmatrix} 10 & 0 & 5 \\ 10 & 5 & 15 \\ 5 & -5 & 0 \end{bmatrix} = \begin{bmatrix} -5 & -1 & -3 \\ -1 & -7 & -10 \\ -5 & 4 & -2 \end{bmatrix}$.

**Q12.** Find : $\displaystyle\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x} dx$.

**Sol.** Put $\tan^3 x = t \Rightarrow \tan^2 x \sec^2 x dx = dt/3$

$\therefore \displaystyle\int \frac{\tan^2 x \sec^2 x}{1 - \tan^6 x} dx = \frac{1}{3}\int \frac{dt}{1 - t^2} = \frac{1}{3} \times \frac{1}{2}\log\left|\frac{1+t}{1-t}\right| + C = \frac{1}{6}\log\left|\frac{1 + \tan^3 x}{1 - \tan^3 x}\right| + C$.

**Q13.** Solve for x : $\tan^{-1}(2x) + \tan^{-1}(3x) = \dfrac{\pi}{4}$.

**Sol.** $\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}$ $\Rightarrow \tan^{-1}\frac{2x+3x}{1-2x.3x}=\frac{\pi}{4}$ $\Rightarrow \frac{5x}{1-6x^2}=\tan\frac{\pi}{4}$

$\Rightarrow 6x^2+5x-1=0$ $\Rightarrow (6x-1)(x+1)=0$ $\Rightarrow x+1=0,\ 6x-1=0$ $\therefore x=-1, 1/6$

But $x=-1$ doesn't satisfy the given equation. So, $x=1/6$ is the required solution.

**Q18.** Using properties of determinants, prove the following :

$$\begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix} = 2(a+b)(b+c)(c+a)$$

**Sol.** LHS : Let $\Delta = \begin{vmatrix} a+b+c & -c & -b \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$ By $R_1 \to R_1+R_2+R_3$

$\Rightarrow \quad = \begin{vmatrix} a & b & c \\ -c & a+b+c & -a \\ -b & -a & a+b+c \end{vmatrix}$ By $R_1 \to R_1-R_2,\ R_2 \to R_2+R_3$

$\Rightarrow \quad = \begin{vmatrix} a+c & -(a+c) & a+c \\ -(b+c) & b+c & b+c \\ -b & -a & a+b+c \end{vmatrix}$

Taking $(a+c)$ & $(b+c)$ common from $R_1$ & $R_2$ respectively.

$\Rightarrow \quad =(a+c)(b+c)\begin{vmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix}$ By $R_1 \to R_1+R_2$

$\Rightarrow \quad =(a+c)(b+c)\begin{vmatrix} 0 & 0 & 2 \\ -1 & 1 & 1 \\ -b & -a & a+b+c \end{vmatrix}$ Expanding along $R_1$

$\therefore \Delta=(a+c)(b+c)\{0-0+2(a+b)\}=2(a+b)(b+c)(c+a)=\text{RHS}.$

**Q19.** If $x=\cos t+\log\tan\left(\frac{t}{2}\right)$, $y=\sin t$, then find the value of $\frac{d^2y}{dt^2}$ and $\frac{d^2y}{dx^2}$ at $t=\frac{\pi}{4}$.

**Sol.** We have $x=\cos t+\log\tan\left(\frac{t}{2}\right)$, $y=\sin t$

Diff. w. r. t. t both the sides, we get : $\frac{dx}{dt}=-\sin t+\frac{1}{\tan\left(\frac{t}{2}\right)}\times\sec^2\left(\frac{t}{2}\right)\times\frac{1}{2}$, $\frac{dy}{dt}=\cos t$

$\Rightarrow \frac{dx}{dt}=-\sin t+\frac{1}{2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}=-\sin t+\frac{1}{\sin t}=\left\{\frac{1-\sin^2 t}{\sin t}\right\}=\cot t\cos t$, $\frac{d^2y}{dt^2}=-\sin t$

So, $\left.\frac{d^2y}{dt^2}\right]_{\text{at } t=\pi/4}=-\sin\frac{\pi}{4}=-\frac{1}{\sqrt{2}}$.

Also, $\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\left(\frac{dt}{dx}\right)=\cos t\times\frac{1}{\cot t\cos t}=\tan t$

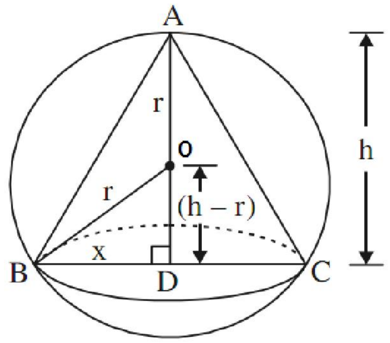
Differentiating w. r. t. x both sides, $\frac{d^2y}{dx^2}=\sec^2 t\times\frac{dt}{dx}=\sec^2 t\times\frac{1}{\cot t\cos t}=\tan t\sec^3 t$

So, $\left.\dfrac{d^2y}{dx^2}\right]_{at\ t=\pi/4} = \tan\dfrac{\pi}{4}\sec^3\dfrac{\pi}{4} = 1\times\left\{\sqrt{2}\right\}^3 = 2\sqrt{2}$.

**Q24.** Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\dfrac{4r}{3}$. Also find the maximum volume of cone.

**Sol.** Let ABC be a cone of maximum volume inscribed in a sphere of radius r.
Let BD = x. In $\Delta OBD$, $OD^2 + BD^2 = OB^2$ $\therefore (h-r)^2 + x^2 = r^2 \Rightarrow x^2 = r^2 - (h-r)^2$

Then volume of cone, $V = \dfrac{1}{3}\pi(BD)^2(AD) \Rightarrow V = \dfrac{1}{3}\pi(x^2)(h) \quad \therefore V = \dfrac{\pi}{3}(2hr - h^2)(h)$




$\Rightarrow V = \dfrac{\pi}{3}(2h^2r - h^3) \qquad \Rightarrow \dfrac{dV}{dh} = \dfrac{\pi}{3}(4hr - 3h^2)$

and, $\dfrac{d^2V}{dh^2} = \dfrac{\pi}{3}\left[4r - 6h\right]$

For $\dfrac{dV}{dh} = 0$, $\dfrac{\pi h}{3}(4r - 3h) = 0 \Rightarrow h = \dfrac{4r}{3}$

As $\left.\dfrac{d^2V}{dh^2}\right]_{at\ h=4r/3} = \dfrac{\pi}{3}\left[4r - 8r\right] = -\dfrac{4\pi r}{3} < 0$

So, V is maximum at $h = \dfrac{4r}{3}$.

Therefore, the altitude of a right circular cone of maximum volume that can be inscribed in a sphere of radius r is 4r/3.

Also, volume of the cone, $V = \dfrac{\pi}{3}(2r - h)h^2 = \dfrac{\pi}{3}\left(2r - \dfrac{4r}{3}\right)\times\left(\dfrac{4r}{3}\right)^2 = \dfrac{\pi}{3}\left(\dfrac{2r}{3}\right)\times\left(\dfrac{16r^2}{9}\right) = \dfrac{32\pi r^3}{81}$.

**Q25.** If $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}$, then find $A^{-1}$. Hence, solve the following system of equations :

$2x - 3y + 5z = 11,\ 3x + 2y - 4z = -5,\ x + y - 2z = -3$.

**OR** Obtain the inverse of the following matrix using elementary operations :

$$A = \begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}.$$

**Sol.** We've $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix} \Rightarrow |A| = \begin{vmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{vmatrix} = 2\times0 + 3\times(-2) + 5\times1 = -1 \neq 0 \ \therefore A^{-1}$ exists.

Consider $A_{ij}$ be the cofactor of $a_{ij}$.

$A_{11} = 0,\ A_{12} = 2,\ A_{13} = 1,$
$A_{21} = 1,\ A_{22} = -9,\ A_{23} = -5,$ $\qquad \Rightarrow adj.A = \begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix}$
$A_{31} = 2,\ A_{32} = 23,\ A_{33} = 13$

$\therefore A^{-1} = \dfrac{1}{-1}\begin{bmatrix} 0 & -1 & 2 \\ 2 & -9 & 23 \\ 1 & -5 & 13 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix} \qquad \left\{\because A^{-1} = \dfrac{1}{|A|}adj.A\right.$

Consider the given systems of equations : $2x - 3y + 5z = 11,\ 3x + 2y - 4z = -5,\ x + y - 2z = -3$

These equations can be expressed as : $AX = B$ where $A = \begin{bmatrix} 2 & -3 & 5 \\ 3 & 2 & -4 \\ 1 & 1 & -2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix}$

Therefore, $X = A^{-1}B \quad \Rightarrow X = \begin{bmatrix} 0 & 1 & -2 \\ -2 & 9 & -23 \\ -1 & 5 & -13 \end{bmatrix}\begin{bmatrix} 11 \\ -5 \\ -3 \end{bmatrix} \quad \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

By equality of matrices, we get : $x = 1, y = 2, z = 3$.

**OR** Using elementary operations, we have $A = AI$ i.e., $\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_1 \leftrightarrow R_2$, $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 2 \\ 3 & 1 & 1 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

By $R_2 \to R_2 + R_1$, $R_3 \to R_3 - 3R_1$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 5 \\ 0 & -5 & -8 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

By $R_2 \to \frac{1}{3}R_2$, $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{5}{3} \\ 0 & -5 & -8 \end{bmatrix} = A\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & -3 & 1 \end{bmatrix}$

By $R_1 \to R_1 - 2R_2$, $R_3 \to R_3 + 5R_2$ $\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & \frac{1}{3} \end{bmatrix} = A\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ \frac{5}{3} & -\frac{4}{3} & 1 \end{bmatrix}$

By $R_3 \to 3R_3$, $\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & \frac{5}{3} \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 \\ 5 & -4 & 3 \end{bmatrix}$

By $R_2 \to R_2 - \frac{5}{3}R_3$, $\begin{bmatrix} 1 & 0 & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} -\frac{2}{3} & \frac{1}{3} & 0 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$

By $R_1 \to R_1 + \frac{1}{3}R_3$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A\begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$

Since $I = AA^{-1}$ $\quad \therefore A^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ -8 & 7 & -5 \\ 5 & -4 & 3 \end{bmatrix}$.

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**Note** that only those questions from set 3 have been given which weren't in set 1 and set 2.

**Q01.** If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$, then find the matrix A.

**Sol.** $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ $\Rightarrow 3A - B + B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 3 & 6 \end{bmatrix}$ $\therefore A = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$.

**Q02.** Write the order and the degree of the following differential equation :

$$x^3\left(\frac{d^2y}{dx^2}\right)^2 + x\left(\frac{dy}{dx}\right)^4 = 0.$$

**Sol.** Order is 2 and degree is also 2.

**Q05.** Find : $\int \sin x.\log\cos x\, dx$.

**Sol.** $\int \sin x.\log\cos x\, dx = \log\cos x \int \sin x\, dx - \int\left(\frac{d}{dx}(\log\cos x)\int \sin x\, dx\right)dx$

$$= -\cos x\log\cos x - \int\left(\frac{\sin x}{\cos x}\times\cos x\right)dx = -\cos x\log\cos x + \cos x + C.$$

**Q06.** Evaluate : $\int_{-\pi}^{\pi}(1-x^2)\sin x\cos^2 x dx$. **OR** Evaluate : $\int_{-1}^{2}\frac{|x|}{x}dx$.

**Sol.** Let $f(x) = (1-x^2)\sin x\cos^2 x$

$\Rightarrow f(-x) = (1-(-x)^2)\sin(-x)\cos^2(-x) = (1-x^2)(-\sin x)\cos^2 x = -(1-x^2)\sin x\cos^2 x = -f(x)$

That means, $f(x)$ is an odd function.

Therefore by using $\int_{-a}^{a} f(x)dx = \begin{cases} 2\int_0^a f(x)dx, \text{ if } f(x) \text{ is even function} \\ 0, \text{ if } f(x) \text{ is odd function} \end{cases}$, we get :

$\int_{-\pi}^{\pi}(1-x^2)\sin x\cos^2 x dx = 0$.

**OR** Let $I = \int_{-1}^{2}\frac{|x|}{x}dx = \int_{-1}^{0}\frac{|x|}{x}dx + \int_{0}^{2}\frac{|x|}{x}dx = \int_{-1}^{0}\frac{-x}{x}dx + \int_{0}^{2}\frac{x}{x}dx = \int_{-1}^{0}-1dx + \int_{0}^{2}1dx$

$\Rightarrow I = -[x]_{-1}^{0} + [x]_0^2 = -(0-(-1)) + 2 - 0 = 1$.

**Q13.** Using properties of determinants, prove the following :

$$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3 + b^3 + c^3 - 3abc.$$

**Sol.** LHS : Let $\Delta = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$ By $C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \Delta = \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix}$$ Taking $(a+b+c)$ common from $C_1$

$$\Rightarrow \Delta = (a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$ Expanding along $C_1$

$$\Rightarrow \Delta = (a+b+c)\left\{(b-c)(a+b)-(c^2-a^2)+2\left(bc-ba-bc+c^2\right)\right\}$$

$$\Rightarrow \Delta = (a+b+c)\left\{ba+b^2-ca-cb-c^2+a^2+2c^2-2ba\right\}$$

$$\Rightarrow \Delta = (a+b+c)\left\{a^2+b^2+c^2-ab-bc-ca\right\}$$

$$\Rightarrow \Delta = a^3+b^3+c^3-3abc = \text{RHS}.$$

**Q20.** Find : $\int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$.

**Sol.** Let $I = \int \frac{\cos x}{(1+\sin x)(2+\sin x)}dx$ Put $1+\sin x = t \Rightarrow \cos x dx = dt$

$$\therefore I = \int \frac{dt}{t(1+t)} = \int\left(\frac{1}{t}-\frac{1}{1+t}\right)dt = \log|t| - \log|1+t| + C = \log\left|\frac{1+\sin x}{2+\sin x}\right| + C.$$

Here we've used $\frac{1}{t(1+t)} = \frac{A}{t}+\frac{B}{1+t} \Rightarrow 1 = A(1+t)+Bt \therefore A=1, B=-1$. So, $\frac{1}{t(1+t)} = \frac{1}{t}-\frac{1}{1+t}$.

**Q21.** Solve the differential equation : $\frac{dy}{dx}-\frac{2x}{1+x^2}y = x^2+2$.

**OR** Solve the differential equation : $(x+1)\frac{dy}{dx} = 2e^{-y}-1;\ y(0)=0$.

**Sol.** We have $\frac{dy}{dx}-\frac{2x}{1+x^2}y = x^2+2$

This is linear diff. eq. of the form $\frac{dy}{dx}+yP(x) = Q(x)$, where $P(x) = -\frac{2x}{1+x^2}$, $Q(x) = x^2+2$.

Now Integrating Factor, $\text{I.F.} = e^{\int -\frac{2x}{1+x^2}dx} = e^{-\log(1+x^2)} = \frac{1}{1+x^2}$

So the solution is given by : $y\times\frac{1}{1+x^2} = \int\frac{x^2+2}{1+x^2}dx + C \quad \Rightarrow \frac{y}{1+x^2} = \int \frac{1+x^2+1}{1+x^2}dx + C$

$$\Rightarrow \frac{y}{1+x^2} = \int\left(1+\frac{1}{1+x^2}\right)dx + C \qquad \Rightarrow \frac{y}{1+x^2} = x+\tan^{-1}x + C$$

Hence $\frac{y}{1+x^2} = x+\tan^{-1}x+C$ or, $y = (x+\tan^{-1}x+C)(1+x^2)$ is the required solution.

**OR** $(x+1)\frac{dy}{dx} = 2e^{-y}-1 \qquad \Rightarrow \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1} \qquad \Rightarrow -\int\frac{-e^y dy}{2-e^y} = \int\frac{dx}{x+1}$

In the first integral, put $2-e^y = t \Rightarrow -e^y dy = dt \qquad \therefore -\int\frac{dt}{t} = \int\frac{dx}{x+1}$

$$\Rightarrow -\log|t| = \log|x+1| + \log C \qquad \Rightarrow -\log\left|2-e^y\right| = \log|C(x+1)|$$

$$\Rightarrow \frac{1}{2-e^y} = \pm C(x+1) \qquad \Rightarrow \frac{1}{2-e^y} = k(x+1), \text{ where } k = \pm C$$

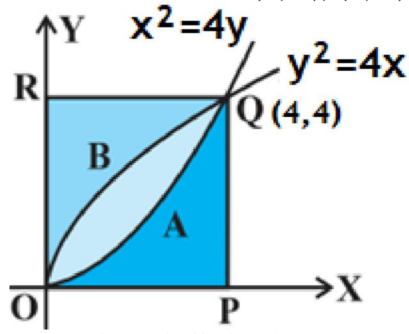
As $y(0)=0$ so, $\frac{1}{2-e^0} = k(0+1) \Rightarrow k = 1$.

Hence, the solution is $\dfrac{1}{2-e^y}=x+1$ or, $x=\dfrac{e^y-1}{2-e^y}$.

**Q26.** Prove that the curves $y^2=4x$ and $x^2=4y$ divide the area of the square bounded by $x=0$, $x=4$, $y=4$ and $y=0$ into three equal parts.

**OR** Using integration, find the area of the triangle, whose vertices are (2, 3), (3, 5) and (4, 4)

**Sol.** Let $y^2=4x \ldots$(i) and $x^2=4y \ldots$(ii)




On solving (i) & (ii) we get : $\left(\dfrac{x^2}{4}\right)^2=4x$

$\Rightarrow x^4-64x=0 \qquad \Rightarrow x(x^3-64)=0$

Either $x=0$ or, $x^3-64=0 \quad \Rightarrow x=0,\ x=4$

$\therefore$ the points of intersection are (0, 0) and (4, 4).

Now, the area of the region OAQBO bounded by the curves (i) and (ii) is given as

$$=\int_0^4\left(2\sqrt{x}-\frac{x^2}{4}\right)dx$$

$$\Rightarrow \quad =\left[\frac{4}{3}x^{3/2}-\frac{x^3}{12}\right]_0^4=\frac{32}{3}-\frac{16}{3}=\frac{16}{3}\text{ Sq.units} \ldots\text{(A)}$$

Again, the area of the region OPQAO bounded by the curves (ii), x = 4 and x-axis (y = 0) is given as $=\dfrac{1}{4}\displaystyle\int_0^4 x^2dx=\frac{1}{4}\left[\frac{x^3}{3}\right]_0^4=\frac{1}{12}(64-0)=\frac{16}{3}$ Sq.units …(B)

Similarly, the area of the region OBQRO bounded by the curve (i), y-axis (x = 0) and y = 4 is given as $=\displaystyle\int_0^4\frac{y^2}{4}dy=\frac{1}{4}\left[\frac{y^3}{3}\right]_0^4=\frac{1}{12}[64-0]=\frac{16}{3}$ Sq.units …(C)

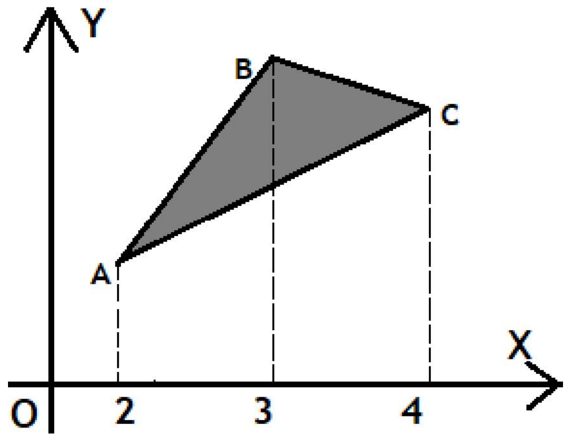
From (A), (B) and (C), it can be concluded that the area of the region OAQBO = area of the region OPQAO = area of the region OBQRO, i.e., area bounded by parabolas (i) and (ii) divides the area of the square in three equal parts.

**OR** Let A(2, 3), B(3, 5) and C(4, 4).

Eq. of AB : y = 2x – 1

Eq. of BC : y = 8 – x

Eq. of CA : $y=\dfrac{x+4}{2}$




Required area is given as

$$\Delta=\int_2^3(2x-1)dx+\int_3^4(8-x)dx-\frac{1}{2}\int_2^4(x+4)dx$$

$$\Rightarrow \Delta=\left[\frac{(2x-1)^2}{2\times 2}\right]_2^3+\left[\frac{(8-x)^2}{2(-1)}\right]_3^4-\frac{1}{2}\left[\frac{(x+4)^2}{2}\right]_2^4$$

$$\Rightarrow \Delta=\frac{1}{4}[25-9]-\frac{1}{2}[16-25]-\frac{1}{4}[64-36]$$

$$\Rightarrow \Delta=4+\frac{9}{2}-7$$

$$\therefore \Delta=\frac{3}{2}\text{ Sq.units}.$$

**Q29.** Two cards are drawn simultaneously (or successively without replacement) from a well shuffled pack of 52 cards. Find the mean and variance of the number of kings.

**Sol.** Let X : Number of kings. So, X can take values 0, 1, 2.

The probability distribution is as follow :

| X | 0 | 1 | 2 |
|---|---|---|---|
| P(X) | $\frac{^{48}C_2}{^{52}C_2}=\frac{188}{221}$ | $\frac{^{4}C_1\times{}^{48}C_1}{^{52}C_2}=\frac{32}{221}$ | $\frac{^{4}C_2}{^{52}C_2}=\frac{1}{221}$ |

Now, $\text{mean}=\sum X\ P(X)=0\times\frac{188}{221}+1\times\frac{32}{221}+2\times\frac{1}{221}=\frac{34}{221}=\frac{2}{13}$

And, $\text{Variance}=\sum X^2\ P(X)-(\text{Mean})^2=0^2\times\frac{188}{221}+1^2\times\frac{32}{221}+2^2\times\frac{1}{221}-\frac{4}{169}=\frac{36}{221}-\frac{4}{169}$

$$=\frac{4}{13}\left(\frac{9}{17}-\frac{1}{13}\right)=\frac{400}{2873}.$$