Series : PTS/16

Code No. 18/12/16

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Candidates must write the Code on the title page of the answer-book.

PLEASURE TEST SERIES XII – 16

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Time Allowed : 180 Minutes **Max. Marks : 100**

SECTION A

Q01. State the reason for the following Binary Operation *, defined on the set Z of integers, to be not commutative : $a*b = ab^3$.

Q02. Let $A=[a_{ij}]_{2\times3}$ and $B=[b_{ij}]_{3\times5}$, write the order of $(AB)^T$.

OR Find the sum of the cofactors of a_{12} and a_{21} in $\begin{vmatrix}2 & 3 & 5\\ 6 & 0 & 4\\ 1 & 5 & -7\end{vmatrix}$.

Q03. Find the approximate percentage increase in the area of a circle if its radius is increased by 2%.

Q04. If the lines $\frac{x-1}{-2}=\frac{y-4}{3p}=\frac{z-3}{4}$ and $\frac{x-2}{4p}=\frac{y-5}{2}=\frac{z-1}{-7}$ are perpendicular to each other then, find the value of p.

SECTION B

Q05. The position vectors of points A and B are $\vec{a}$ and $\vec{b}$ respectively. P divides AB in the ratio of 3 : 1 and Q is mid-point of AP. Find the position vector of Q.

Q06. Find the value of $|\lambda\ \text{adj}.A|$, if $A = diag\begin{pmatrix}a & b & c\end{pmatrix}$.

OR Find the sum of x and y in the matrix equation : $\begin{pmatrix}3\\2\end{pmatrix}\begin{pmatrix}1 & 5\end{pmatrix}=\begin{pmatrix}3 & 7x+y\\ 2y & 10\end{pmatrix}$.

Q07. Find $\int e^x(\cos x-\sin x)dx$.

Q08. If $|\vec{a}|=2, |\vec{b}|=2\sqrt{3}$ and $\vec{a}\perp\vec{b}$, then write the value of $|\vec{a}+\vec{b}|$.

OR Using vectors, check if (3,–5, 1), (–1, 0, 8) and (7,–10,–6) are collinear points or not.

Q09. Evaluate : $\int_{-\pi/4}^{\pi/4} x^3\sin^4 x\,dx$. **OR** Find $\int y^{-1}\ x!\ dy,\ x\in Z^+\cup\{0\}$.

Q10. Show that $xdy=[y+\sqrt{x^2+y^2}]dx,\ x>0$ is homogenous differential equation.

Q11. Find $\frac{dy}{dx}$, if $y=\frac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}}+\frac{1}{1+x^{\beta-\gamma}+x^{\alpha-\gamma}}+\frac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}}$, where $\alpha,\beta,\gamma\in R$.

Q12. If A and B are two events such that $A\subset B$ and $B\neq\phi$, then what is the relation between $P(A|B)$ and $P(A)$?

SECTION C

Q13. Let $A=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}4 & -6\\ -2 & 4\end{bmatrix}$. Then compute AB. Hence, solve the following system of equations : $2x+y=4,\ 3x+2y=1$.

Q14. Simplify $2\tan^{-1}(\text{cosec}\tan^{-1}x-\tan\cot^{-1}x)$.

Q15. If the function $f(x)=\begin{cases}x^2, \text{ if } x\le 2\\ ax+b, \text{ if } x>2\end{cases}$ is differentiable at x = 2, then find the value of a and b.

Q16. Let $x=a\sin pt$ and, $y=b\cos pt$. Then find second order derivative of y w.r.t. x at $t=0$.

Q17. Show that the condition that curves $ax^2 + by^2 = 1$ and $mx^2 + ny^2 = 1$ should intersect each other orthogonally, is that $a^{-1} - b^{-1} = m^{-1} - n^{-1}$.

OR The total area of a page is 150cm^2. The combined width of the margin at the top and bottom is 3cm and the side 2cm. What must be the dimensions of the page in order that the area of the printed matter may be maximum?

Q18. Evaluate : $\int (\sec x + \cos x)^{-1} dx$.

Q19. If $\vec{a}, \vec{b}, \vec{c}$ are three mutually perpendicular vectors of equal magnitude, prove that $\vec{a} + \vec{b} + \vec{c}$ is equally inclined with vectors $\vec{a}, \vec{b}$ and, $\vec{c}$.

OR Points L, M, N divide the sides BC, CA and AB of ΔABC in the ratio 1:4, 3:2, 3:7 respectively. Prove that $\overrightarrow{AL} + \overrightarrow{BM} + \overrightarrow{CN}$ is a vector parallel to $\overrightarrow{CK}$, where K divides the side AB in the ratio 1:3.

Q20. Solve : $xdy + (y - x + xy\cot x)dx = 0;\ x \neq 0$.

Q21. Find the equation of a line passing through the point (1, 2, –4) and perpendicular to two lines $\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$ and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

OR A line with d.r.'s 2, 1, 2 meets each of the lines $x = y + a = z$ and $x + a = 2y = 2z$. Find the coordinates of the point of intersection.

Q22. A bag contains 25 balls of which 10 balls bear a mark 'A' and the remaining balls bear a mark 'B'. A ball is drawn, its mark is noted and is replaced. If 6 balls are drawn in this way, find the probability that

(a) all will bear mark 'A' (b) not more than 2 will bear mark 'B'

(c) at least one ball will bear 'B' (d) the number of balls with mark 'A' & 'B' will be equal.

Q23. A problem in Mathematics is given to 4 students A, B, C and D. Their chances of solving the problems respectively are 1/3, 1/4, 1/5 and 2/3. What is the probability that (i) the problem will be solved? (ii) at most one of them will solve the problem?

SECTION D

Q24. Let a relation R on set of natural numbers N be defined as $(x, y) \in R \Leftrightarrow x^2 - 4xy + 3y^2 = 0\ \forall\ x, y \in N$. Verify that R is reflexive but not symmetric and transitive.

OR Let X is a non-empty set and * : P(X) $\times$P(X) $\rightarrow$P(X) be defined as A*B = (A – B) $\cup$ (B – A) $\forall$ A, B$\in$P(X). Show that ϕ is the identity for the operation * and all the elements A of P(X) are invertible with A^{-1}.

Q25. Let A and B be a 3×3 matrix. Let $A + B = 2B^T$ and $3A + 2B = I_3$. Determine the value of the matrix sum $10A + 5B$.

OR If $p + q + r = 0,\ a + b + c = 0$, then show that $\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix} = 0$.

Q26. Find the area bounded by the curve $y = \frac{x^2}{4} + \frac{x}{4} - \frac{1}{2}$ with x-axis in [0, 2].

Q27. Evaluate : $\int_{-\pi}^{\pi} \frac{2x(1 + \sin x)}{1 + \cos^2 x} dx$. **OR** Evaluate : $\int x(\cot^{-1} x)^2 dx$.

Q28. Find the equation of the plane passing through the line of intersection of the planes $\vec{r}.(2\hat{i} + 3\hat{j} - \hat{k}) = -1$ and $\vec{r}.(\hat{i} + \hat{j} - 2\hat{k}) = 0$ and passing through the point (3, –2, –1). Also find the angle between the two given planes.

Q29. A library has to accommodate two different types of books of Maths - one on Algebra and other on Calculus - on a shelf. The Algebra books are 6 cm and the books on Calculus are 4 cm thick and they weigh 1 kg and 1.5 kg respectively. The shelf is 90 cm long and at most can support a weight of 21 kg. How the shelf should be filled with the books of two types in order to include maximum number of books? Form an LPP and solve it graphically.

ANSWERS & HINTS For PTS-16

SECTION A

Q01. Since $1*2 = 1\times 2^3 = 8$ but $2*1 = 2\times 1^3 = 2 \neq 1*2$ where $1, 2 \in Z$. So * isn't commutative.

Q02. As order of AB is 2×5 so, clearly order of $(AB)^T$ is 5×2.

OR Cofactor of $a_{12} = -[(6)(-7)-(4)(1)] = 46$ and, cofactor of $a_{21} = -[(3)(-7)-(5)(5)] = 46$. Therefore, the required sum is 92.

Q03. Here $\left(\frac{\Delta r}{r}\times 100\right)\% = 2\% \Rightarrow \frac{\Delta r}{r} = 0.02 \Rightarrow \Delta r = 0.02r \approx dr$

Now area, $A = \pi r^2 \Rightarrow \frac{dA}{dr} = 2\pi r \Rightarrow \frac{dA}{A} = \frac{2\pi r\, dr}{\pi r^2} \Rightarrow \frac{dA}{A} = \frac{2\times 0.02\, r}{r}$

$\Rightarrow \left(\frac{dA}{A}\times 100\right)\% = (0.04\times 100)\% = 4\%$.

Q04. Use $a_1a_2 + b_1b_2 + c_1c_2 = 0$ for $\perp^{er}$ lines to get $p = -14$.

SECTION B

Q05. Here $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}$. As P divides AB in the ratio of 3 : 1 so, $\overrightarrow{OP} = \frac{3\vec{b}+\vec{a}}{3+1} = \frac{3\vec{b}+\vec{a}}{4}$.

Also Q is the midpoint of AP. So, $\overrightarrow{OQ} = \frac{\overrightarrow{OA}+\overrightarrow{OP}}{2} = \frac{\vec{a}+\frac{3\vec{b}+\vec{a}}{4}}{2} = \frac{5\vec{a}+3\vec{b}}{8}$.

Q06. As $|\lambda\ adj.A| = \lambda^3|adj.A| = \lambda^3|A|^{3-1} = \lambda^3a^2b^2c^2$ $[\because |A| = abc$

OR $\begin{pmatrix}3\\2\end{pmatrix}\begin{pmatrix}1 & 5\end{pmatrix} = \begin{pmatrix}3 & 7x+y\\ 2y & 10\end{pmatrix} \Rightarrow \begin{pmatrix}3 & 15\\ 2 & 10\end{pmatrix} = \begin{pmatrix}3 & 7x+y\\ 2y & 10\end{pmatrix}$

By equality of matrices, $7x+y = 15,\ 2 = 2y \Rightarrow y = 1,\ x = 2\ \therefore x+y = 3$.

Q07. $e^x\cos x + c$.

Q08. Let $y = |\vec{a}+\vec{b}|$ $\Rightarrow y^2 = |\vec{a}+\vec{b}|^2$ $\Rightarrow y^2 = (\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

$\Rightarrow y^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b}$ $\Rightarrow y^2 = 4+12+2\times 0$ [since $\vec{a}\perp\vec{b}$ so, $\vec{a}.\vec{b} = 0$

$\Rightarrow y = 4$ $\therefore |\vec{a}+\vec{b}| = 4$.

OR Let A(3,–5, 1), B(–1, 0, 8) and C(7,–10,–6). Show that the d.r.'s of AB and BC are proportional, which implies $AB \parallel BC$. But B is common point so, A, B and C are collinear.

Q09. Let $f(x) = x^3\sin^4 x$. As $f(-x) = (-x)^3\sin^4(-x) = -[x^3\sin^4 x] = -f(x)$.

So using $\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is an odd function, we have $\int_{-\pi/4}^{\pi/4} x^3\sin^4 x\,dx = 0$.

OR Let $I = \int \frac{x!}{y}dy,\ x \in Z^+ \cup \{0\}$

$\therefore I = x!\,[\,\log|y|\,] + C$ where C is constant of integration, $x \in Z^+ \cup \{0\}$.

Q10. We have $xdy = [\,y + \sqrt{x^2+y^2}\,]dx$ i.e., $\frac{dy}{dx} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x},\ x > 0$.

Consider $f(x,y) = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x}$. Put $x = \lambda x,\ y = \lambda y$

$\Rightarrow f(\lambda x, \lambda y) = \frac{\lambda y}{\lambda x} + \frac{\sqrt{\lambda^2x^2+\lambda^2y^2}}{\lambda x} = \frac{y}{x} + \frac{\sqrt{x^2+y^2}}{x} = f(x,y)$. So, f is homogeneous.

Q11. Here $y=\frac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}}+\frac{1}{1+x^{\beta-\gamma}+x^{\alpha-\gamma}}+\frac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}}$

$\Rightarrow y=\frac{x^{\beta}}{x^{\beta}+x^{\alpha}+x^{\gamma}}+\frac{x^{\gamma}}{x^{\gamma}+x^{\beta}+x^{\alpha}}+\frac{x^{\alpha}}{x^{\alpha}+x^{\beta}+x^{\gamma}}=1 \qquad \therefore \frac{dy}{dx}=0$.

Q12. As $A\subset B\ \therefore A\cap B=A\ \Rightarrow P(A\cap B)=P(A)$ and, $P(A)<P(B)$.

Consider $P(A\mid B)=\frac{P(A\cap B)}{P(B)}=\frac{P(A)}{P(B)}$...(i)

As we know that $P(B)\le 1\ \Rightarrow \frac{1}{P(B)}\ge 1 \qquad \Rightarrow \frac{P(A)}{P(B)}\ge P(A)$...(ii)

By (i) and (ii), $P(A\mid B)\ge P(A)$.

SECTION C

Q13. Here $A=\begin{bmatrix}2&3\\1&2\end{bmatrix}$ and $B=\begin{bmatrix}4&-6\\-2&4\end{bmatrix} \qquad \Rightarrow AB=\begin{bmatrix}2&3\\1&2\end{bmatrix}\begin{bmatrix}4&-6\\-2&4\end{bmatrix}=\begin{bmatrix}2&0\\0&2\end{bmatrix}=2\,I$

$\Rightarrow A\left(\frac{1}{2}B\right)=I\ \therefore A^{-1}=\frac{1}{2}B$...(i)

The given system of equations $2x+y=4,\ 3x+2y=1$ can be written as

$PX=C$ where, $P=\begin{bmatrix}2&1\\3&2\end{bmatrix}$, $X=\begin{bmatrix}x\\y\end{bmatrix}$ and $C=\begin{bmatrix}4\\1\end{bmatrix}$

$\Rightarrow P^{-1}PX=P^{-1}C \qquad \Rightarrow IX=P^{-1}C \qquad \therefore X=P^{-1}C$ [Note that $P=A^{T}\ \therefore P^{-1}=(A^{-1})^{T}$

Therefore, $X=\frac{1}{2}\begin{bmatrix}4&-2\\-6&4\end{bmatrix}\begin{bmatrix}4\\1\end{bmatrix} \qquad \Rightarrow X=\frac{1}{2}\begin{bmatrix}14\\-20\end{bmatrix} \qquad \Rightarrow \begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}7\\-10\end{bmatrix}$

By equality of matrices, we get : $x=7,\ y=-10$.

Q14. See **O.P. Gupta's Mathematicia**. Ans. $\tan^{-1}x$.

Q15. As the function f is differentiable at x = 2, so it is continuous at x = 2 as well.

$\therefore \lim_{x\to 2^-} f(x)=\lim_{x\to 2^+} f(x)=f(2) \qquad \Rightarrow \lim_{x\to 2^-} x^2=\lim_{x\to 2^+} ax+b=(2)^2\ \Rightarrow 4=2a+b$ ...(i)

Also, f is differentiable at x = 2 $\quad \therefore Lf'(2)=Rf'(2)$ i.e., $\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}$

$\Rightarrow \lim_{x\to 2^-}\frac{x^2-4}{x-2}=\lim_{x\to 2^+}\frac{(ax+b)-4}{x-2} \qquad \Rightarrow \lim_{x\to 2^-}(x+2)=\lim_{x\to 2^+}\frac{(ax+b)-4}{x-2}$ [by (i), $b=4-2a$

$\therefore 4=\lim_{x\to 2^+}\frac{(ax+4-2a)-4}{x-2} \qquad \Rightarrow 4=\lim_{x\to 2^+}\frac{(x-2)a}{x-2}=\lim_{x\to 2^+} a \qquad \Rightarrow a=4$

Replacing value of a in (i), we get : $b=-4$.

Q16. We have $x=a\sin pt$ and, $y=b\cos pt \qquad \Rightarrow \frac{dx}{dt}=ap\cos pt$ and, $\frac{dy}{dt}=-bp\sin pt$

$\because \frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx} \qquad \Rightarrow \frac{dy}{dx}=-\frac{b\sin pt}{a\cos pt}=-\frac{b}{a}\tan pt$

$\therefore \frac{d^2y}{dx^2}=-\frac{bp}{a}\sec^2 pt\times\frac{dt}{dx}=-\frac{bp}{a}\sec^2 pt\times\frac{1}{ap\cos pt}=-\frac{b}{a^2}\sec^3 pt$

Therefore, $\left.\frac{d^2y}{dx^2}\right]_{\text{at } t=0}=-\frac{b}{a^2}\sec^3 0=-\frac{b}{a^2}$.

Q17. Given curves $ax^2+by^2=1$...(i) and $mx^2+ny^2=1$ ...(ii)

$\therefore 2ax+2by\frac{dy}{dx}=0$ and $2mx+2ny\frac{dy}{dx}=0 \qquad \Rightarrow \frac{dy}{dx}=-\frac{ax}{by}=m_1$ and $\frac{dy}{dx}=-\frac{mx}{ny}=m_2$

If (i) & (ii) intersect orthogonally then, $m_1 m_2 = -1$ $\Rightarrow \left(-\dfrac{ax}{by}\right)\left(-\dfrac{mx}{ny}\right) = -1$

$\Rightarrow \dfrac{amx^2}{bny^2} = -1$ $\Rightarrow \dfrac{x^2}{y^2} = -\dfrac{bn}{am} \ldots$(iii)

Subtracting (ii) from (i), $(a-m)x^2 + (b-n)y^2 = 0$ $\Rightarrow \dfrac{x^2}{y^2} = -\dfrac{b-n}{a-m} \ldots$(iv)

By (iii) & (iv), $-\dfrac{bn}{am} = -\dfrac{b-n}{a-m}$ $\Rightarrow \dfrac{a}{am} - \dfrac{m}{am} = \dfrac{b}{bn} - \dfrac{n}{bn}$

$\Rightarrow \dfrac{1}{m} - \dfrac{1}{a} = \dfrac{1}{n} - \dfrac{1}{b}$ $\therefore \dfrac{1}{a} - \dfrac{1}{b} = \dfrac{1}{m} - \dfrac{1}{n}$.

OR Ans. $15 \times 10\ cm^2$. See **O.P. Gupta's Mathematicia**.

Q18. See **O.P. Gupta's Mathematicia**. Ans. $\dfrac{1}{2\sqrt{2}} \log \left| \dfrac{\sqrt{2} + \sin x}{\sqrt{2} - \sin x} \right| + C$.

Q19. See **O.P. Gupta's Mathematicia**. **OR** See **O.P. Gupta's Mathematicia**.

Q20. $xy \sin x = \sin x - x \cos x + C$.

Q21. Let the d.r.'s of required line L be a, b, c.

Since L is perpendicular to the lines $\vec{r} = (8\hat{i} - 19\hat{j} + 10\hat{k}) + \lambda(3\hat{i} - 16\hat{j} + 7\hat{k})$ and $\vec{r} = (15\hat{i} + 29\hat{j} + 5\hat{k}) + \mu(3\hat{i} + 8\hat{j} - 5\hat{k})$.

$\left.\begin{array}{l} \therefore\ 3a - 16b + 7c = 0 \\ \text{and } 3a + 8b - 5c = 0 \end{array}\right\}$ On solving these, we get : $\dfrac{a}{24} = \dfrac{b}{36} = \dfrac{c}{72}$ i.e., d.r.'s of L : 2, 3, 6.

Hence required line through (1, 2, –4) is $\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$.

OR We have $L_1 : x = y + a = z \Rightarrow \dfrac{x}{1} = \dfrac{y+a}{1} = \dfrac{z}{1} = \lambda(\text{say})$

and $L_2 : x + a = 2y = 2z \Rightarrow \dfrac{x+a}{1} = \dfrac{y}{1/2} = \dfrac{z}{1/2} \Rightarrow \dfrac{x+a}{2} = \dfrac{y}{1} = \dfrac{z}{1} = \mu(\text{say})$

$\therefore$ the coordinates of any random point on the lines L_1 and L_2 are $P(\lambda, \lambda - a, \lambda)$ and $Q(2\mu - a, \mu, \mu)$ respectively. The direction ratios of line PQ : $2\mu - \lambda - a,\ \mu - \lambda + a,\ \mu - \lambda$

Also it is given that the d.r.'s of line PQ are proportional to 2, 1, 2.

Therefore $\dfrac{2\mu - \lambda - a}{2} = \dfrac{\mu - \lambda + a}{1} = \dfrac{\mu - \lambda}{2}$. Consider $\dfrac{2\mu - \lambda - a}{2} = \dfrac{\mu - \lambda + a}{1}$ and $\dfrac{\mu - \lambda + a}{1} = \dfrac{\mu - \lambda}{2}$.

On solving we get : $\lambda = 3a$ and $\mu = a$.

Hence points of intersection of line PQ with L_1 is P(3a, 2a, 3a) and with L_2 is Q(a, a, a).

Q22. Here n = 6.

(a) Let p = probability of getting balls with mark A = 10/25 = 2/5, q = 3/5.

$\therefore$ Probability of getting all balls with mark A, $P(6) = {}^6C_6 (2/5)^6 (3/5)^{6-6} = (2/5)^6$

(b) Here p = probability of getting balls with mark B = probability of not getting balls with mark A = 15/25 = 3/5, q = 2/5.

So, probability of getting not more than 2 balls with mark B is given by i

$P(x \le 2) = P(0) + P(1) + P(2) = {}^6C_0 (3/5)^0 (2/5)^{6-0} + {}^6C_1 (3/5)^1 (2/5)^{6-1} + {}^6C_2 (3/5)^2 (2/5)^{6-2}$

$\Rightarrow \quad = 7(2/5)^4$

(c) p = probability of getting balls with mark B = 15/25 = 3/5, q = 2/5

$\therefore$ Probability of getting at least one ball with mark B = $P(x \ge 1) = 1 - P(x < 1)$

$\Rightarrow \quad = 1 - {}^6C_0 (3/5)^0 (2/5)^{6-0} = 1 - (2/5)^6$.

(d) p = probability of getting balls with mark A = 10/25 = 2/5, q = 3/5

$\therefore$ Probability of getting same no. of balls with mark A and B $= P(x=3) = {}^6C_3\,(2/5)^3\,(3/5)^{6-3}$

$\Rightarrow \quad = 20(2/5)^3(3/5)^3$.

Q23. Let A, B, C and D denote the events that A, B, C and D solves the problem in Mathematics resp.

$\therefore P(A)=\frac{1}{3}, P(B)=\frac{1}{4}, P(C)=\frac{1}{5}$ and $P(D)=\frac{2}{3} \quad \Rightarrow P(\bar{A})=\frac{2}{3}, P(\bar{B})=\frac{3}{4}, P(\bar{C})=\frac{4}{5}$ and $P(\bar{D})=\frac{1}{3}$.

(i) P(the problem will be solved) $= P(A\cup B\cup C\cup D) = 1-P(\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D})$

$\Rightarrow \quad = 1-P(\bar{A})\times P(\bar{B})\times P(\bar{C})\times P(\bar{D}) = 1-\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{1}{3}=\frac{13}{15}$.

(ii) P(at most one of them will solve the problem)

$$= P(\bar{A}\cap\bar{B}\cap\bar{C}\cap\bar{D}) + P(A\cap\bar{B}\cap\bar{C}\cap\bar{D}) + P(\bar{A}\cap B\cap\bar{C}\cap\bar{D}) + P(\bar{A}\cap\bar{B}\cap C\cap\bar{D}) + P(\bar{A}\cap\bar{B}\cap\bar{C}\cap D)$$

$$\Rightarrow \quad = \frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{1}{3}+\frac{1}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{1}{3}+\frac{2}{3}\times\frac{1}{4}\times\frac{4}{5}\times\frac{1}{3}+\frac{2}{3}\times\frac{3}{4}\times\frac{1}{5}\times\frac{1}{3}+\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{2}{3}=\frac{49}{90}.$$

SECTION D

Q24. Here $(x,y)\in R \Leftrightarrow x^2-4xy+3y^2=0 \ \forall\ x,y\in N$.

Reflexivity : Let $x\in N \ \therefore x^2-4x.x+3x^2=0$. So, $(x,x)\in R \quad \therefore$ R is reflexive .

Symmetry : Let $(3,1)\in R$. Clearly, $3^2-4.3.1+3.1^2=12-12=0$

But $(1,3)\notin R$ as, $1^2-4.1.3+3.3^2=16\neq 0$. So, R isn't symmetric.

Transitivity : Let $(9,3), (3,1)\in R$. Clearly, $9^2-4.9.3+3.3^2=0$ and $3^2-4.3.1+3.1^2=0$.

But $9^2-4.9.1+3.1^2=48\neq 0 \ \therefore (9,1)\notin R$. Hence R isn't transitive.

OR Suppose E be an identity element, then $A*E=E*A=A$ for all $A\in P(X)$

$\Rightarrow (A-E)\cup(E-A)=A$ for all $A\in P(X)$.

Taking $A=\phi$, we have : $(\phi-E)\cup(E-\phi)=\phi \quad \Rightarrow \phi\cup E=\phi \quad \Rightarrow E=\phi$.

So, $A*\phi=\phi*A=(A-\phi)\cup(\phi-A)=A \ \ \forall\ A\in P(X)$.

Hence ϕ is the identity element.

Let $A\in P(X)$ be invertible, then $B\in P(X)$ be inverse of A. Clearly $A*B=B*A=\phi$.

That implies, $(A-B)\cup(B-A)=\phi \quad \Rightarrow (A-B)=\phi$ as well as $(B-A)=\phi$

$\Rightarrow A\subset B$ as well as $B\subset A \quad \therefore A=B$

Thus, for all $A\in P(X)$, $A*A=\phi$. Hence A is invertible and $B=A^{-1}=A$.

Q25. As $A+B=2B^T \qquad \Rightarrow A=2B^T-B$...(i)

Also $3A+2B=I_3 \qquad \Rightarrow 3(2B^T-B)+2B=I_3$ [By using (i)

$\therefore 6B^T-B=I$ …(ii)

Also $[6B^T-B]^T=I^T \quad \Rightarrow 6B-B^T=I$...(iii)

Comparing (ii) and (iii), we get : $6B^T-B=6B-B^T \qquad \Rightarrow 7B^T=7B \quad \therefore B^T=B$

By (i), $A=2B-B=B$ …(iv)

Also, $3A+2B=I_3 \qquad \Rightarrow 3A+2A=I_3 \qquad \therefore 5A=I_3$ [By (iv), A = B

Now $10A+5B=10A+5A=15A=3I_3$ [$\because$ A = B and $5A=I_3$

OR LHS : Let $\Delta=\begin{vmatrix} pa & qb & rc\\ qc & ra & pb\\ rb & pc & qa\end{vmatrix}$ On expanding along R_1,

$\Rightarrow \Delta = pa\left(a^2qr-p^2bc\right)-qb\left(q^2ac-b^2pr\right)+rc\left(c^2pq-r^2ab\right)$

$\Rightarrow \Delta = \left(a^3pqr-p^3abc\right)-\left(q^3abc-b^3pqr\right)+\left(c^3pqr-r^3abc\right)$

$\Rightarrow \Delta = pqr\left(a^3+b^3+c^3\right) - abc\left(p^3+q^3+r^3\right) \ldots(i)$

As $p+q+r=0,\ a+b+c=0\ \ \therefore p^3+q^3+r^3-3pqr=0,\ a^3+b^3+c^3-3abc=0 \ldots(ii)$

By (i) and (ii), $\Delta = pqr(3abc) - abc(3pqr) = 0 = RHS$.

Q26. Ans. $\frac{3}{4}$ Sq.units . See **HOTS IN MATHEMATICS (Level II)**.

Q27. Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)}{1+\cos^2 x}dx$ $\quad\Rightarrow I = \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}dx + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$

Consider $f(x) = \frac{2x}{1+\cos^2 x} \Rightarrow f(-x) = \frac{-2x}{1+\cos^2 x} = -f(x)$

and, $g(x) = \frac{2x\sin x}{1+\cos^2 x} \Rightarrow g(-x) = \frac{2(-x)\sin(-x)}{1+\cos^2(-x)} = \frac{2x\sin x}{1+\cos^2 x} = g(x)$

Clearly, f (x) is an odd function whereas g (x) is an even function.

So, $I = 0 + 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$ $\quad\Rightarrow I = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx \ldots(i)$

$\Rightarrow I = 4\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx$ $\quad\Rightarrow I = 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx \ldots(ii)$

On adding (i) & (ii), we get : $2I = 4\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}dx + 4\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\cos^2 x}dx$

$\Rightarrow 2I = 4\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$ $\quad\Rightarrow I = 2\pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$

Consier $f(x) = \frac{\sin x}{1+\cos^2 x} \Rightarrow f(\pi-x) = \frac{\sin(\pi-x)}{1+\cos^2(\pi-x)} = \frac{\sin x}{1+\cos^2 x} = f(x)$

By using $\int_0^{2a} f(x)dx = \begin{cases} 2\int_0^{a} f(x)dx, \text{ if } f(2a-x) = f(x) \\ 0, \text{ if } f(2a-x) = -f(x) \end{cases}$

We have $I = 2\pi \times 2\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}dx$ $\quad$ (Put $\cos x = t \Rightarrow \sin x dx = -dt$

When $x = 0 \Rightarrow t = 1$ & when $x = \frac{\pi}{2} \Rightarrow t = 0$ $\quad \therefore I = -2\pi \times 2\int_1^{0} \frac{dt}{1+t^2}$

$\Rightarrow I = 4\pi\int_0^{1} \frac{dt}{1+t^2}$ $\quad\Rightarrow I = 4\pi\left[\tan^{-1} t\right]_0^1$ $\quad\Rightarrow I = 4\pi\left[\tan^{-1}1 - \tan^{-1}0\right] = 4\pi\left[\frac{\pi}{4} - 0\right] = \pi^2$.

OR Let $I = \int x(\cot^{-1}x)^2 dx$ $\quad$ (Put $\cot^{-1}x = \theta \Rightarrow \cot\theta = x \Rightarrow dx = -\operatorname{cosec}^2\theta d\theta$

$\therefore I = \int \theta^2[-\cot\theta \operatorname{cosec}^2\theta]d\theta$ $\quad\Rightarrow I = \theta^2\int -\cot\theta \operatorname{cosec}^2\theta d\theta - \int\left(\frac{d}{d\theta}[\theta^2]\int -\cot\theta \operatorname{cosec}^2\theta d\theta\right)d\theta$

$\Rightarrow I = \theta^2\left(\frac{\cot^2\theta}{2}\right) - \int 2\theta\left(\frac{\cot^2\theta}{2}\right)d\theta$ $\quad\Rightarrow I = \theta^2\left(\frac{\cot^2\theta}{2}\right) - \int \theta\left(\operatorname{cosec}^2\theta - 1\right)d\theta$

$\Rightarrow I = \theta^2\left(\frac{\cot^2\theta}{2}\right) - \int \theta \operatorname{cosec}^2\theta d\theta + \int \theta d\theta$

$\Rightarrow I = \theta^2\left(\frac{\cot^2\theta}{2}\right) - \left\{\theta\int \operatorname{cosec}^2\theta d\theta - \int\left(\frac{d}{d\theta}[\theta]\int \operatorname{cosec}^2\theta d\theta\right)d\theta\right\} + \frac{\theta^2}{2}$

$$\Rightarrow I = \theta^2\left(\frac{\cot^2\theta}{2}\right) - \{-\theta\cot\theta + \log|\sin\theta|\} + \frac{\theta^2}{2} + C$$

$$\Rightarrow I = \theta^2\left(\frac{\cot^2\theta}{2}\right) + \theta\cot\theta - \log\left|\frac{1}{\sqrt{1+\cot^2\theta}}\right| + \frac{\theta^2}{2} + C$$

$$\therefore I = \frac{(x\cot^{-1}x)^2}{2} + x\cot^{-1}x + \frac{1}{2}\log|1+x^2| + \frac{(\cot^{-1}x)^2}{2} + C.$$

Q28. Any plane passing through the line of intersection of given planes is

$\vec{r}.(2\hat{i}+3\hat{j}-\hat{k})+1+\lambda\left[\vec{r}.(\hat{i}+\hat{j}-2\hat{k})\right]=0$ i.e., $\vec{r}.\left[(2\hat{i}+3\hat{j}-\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k})\right]+1=0$...(i)

If plane (i) contains (3, –2, –1) then, $(3\hat{i}-2\hat{j}-\hat{k}).\left[(2\hat{i}+3\hat{j}-\hat{k})+\lambda(\hat{i}+\hat{j}-2\hat{k})\right]+1=0$

$\Rightarrow [6-6+1+\lambda(3-2+2)]+1=0 \qquad \Rightarrow \lambda=-2/3$

Replacing value of $\lambda=-2/3$ in (i), we get $\vec{r}.(4\hat{i}+7\hat{j}+\hat{k})+3=0$.

Let θ be the angle between normals of the given planes (as angle between the planes and their normals remains the same).

So, $\cos\theta=\dfrac{(2\hat{i}+3\hat{j}-\hat{k}).(\hat{i}+\hat{j}-2\hat{k})}{\sqrt{4+9+1}\sqrt{1+1+4}}=\dfrac{7}{\sqrt{84}} \qquad \therefore \theta=\cos^{-1}\left(\dfrac{7}{\sqrt{84}}\right)$.

Q29. Let the library has x and y no. of Algebra and Calculus books.

To maximize : Z = x + y

Subject to constraints : $x\ge 0,\ y\ge 0,\ 6x+4y\le 90,\ x+1.5y\le 21$.

Also the shelf can carry a maximum of 10 and 7 books of Algebra and Calculus, respectively.

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