CBSE 2018 ANNUAL EXAMINATION

(Series SGN Code No. 65 (B) : FOR BLIND CANDIDATES)

Max. Marks : 100 **Time Allowed : 3 Hours**

SECTION – A

Q01. If A and B are square matrices, each of order 2 such that $|A| = 3$ and $|B| = -2$, then write the value of $|3AB|$.

Sol. $|3AB| = 3^2|AB| = 9|A||B| = 9\times 3\times(-2) = -54$.

Q02. Write the derivative of $|x - 5|$ at $x = 2$.

Sol. Let $f(x) = |x-5|$ $\quad\therefore f(x) = \begin{cases} x-5, \text{ if } x\ge 5 \\ 5-x, \text{ if } x<5\end{cases}$

As $f(2) = 5-2 = 3$.

Now $Lf'(2) = \lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^-}\frac{(5-x)-3}{x-2} = \lim_{x\to 2^-}\frac{2-x}{x-2} = \lim_{x\to 2^-}(-1) = -1$

And, $Rf'(2) = \lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2} = \lim_{x\to 2^+}\frac{(5-x)-3}{x-2} = \lim_{x\to 2^+}\frac{2-x}{x-2} = \lim_{x\to 2^+}(-1) = -1 = Lf'(2)$.

Hence $f'(2) = -1$.

Alternatively, $f(x) = |x-5|$ $\quad\therefore f(x) = \begin{cases} x-5, \text{ if } x\ge 5 \\ 5-x, \text{ if } x<5\end{cases}$ $\quad\Rightarrow f'(x) = \begin{cases} 1, \text{ if } x\ge 5 \\ -1, \text{ if } x<5\end{cases}$

$\therefore f'(2) = -1$.

Q03. Find : $\int a^x.e^x dx$.

Sol. $\int a^x.e^x dx = \int (ae)^x dx = \frac{(ae)^x}{\log(ae)} + c = \frac{a^x.e^x}{\log a + 1} + c$ $\quad[\because \log e = 1$.

Q04. Find the cosine of the angle which the vector $\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}$ makes with y-axis.

Sol. Let $\vec{a} = \sqrt{2}\,\hat{i} + \hat{j} + \hat{k}$ $\quad\Rightarrow \hat{a} = \frac{\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}}{\sqrt{2+1+1}} = \frac{\sqrt{2}\,\hat{i} + \hat{j} + \hat{k}}{2}$

$\because \hat{a} = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{2} + \frac{\hat{k}}{2} = \cos\alpha\hat{i} + \cos\beta\hat{j} + \cos\gamma\hat{k}$ (On comparing both sides)

$\therefore \cos\beta = \frac{1}{2}$, where β is the angle made by $\vec{a}$ with y-axis.

SECTION – B

Q05. If $\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right) = \frac{\pi}{2}$, find the value of x.

Sol. $\sin^{-1}\left(\frac{\sin x + \cos x}{\sqrt{2}}\right) = \frac{\pi}{2}$ $\quad\Rightarrow \sin^{-1}\left(\sin x\times\frac{1}{\sqrt{2}} + \cos x\times\frac{1}{\sqrt{2}}\right) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\left(\sin\left(x+\frac{\pi}{4}\right)\right) = \frac{\pi}{2}$ $\quad\Rightarrow x + \frac{\pi}{4} = \frac{\pi}{2}$ $\quad\therefore x = \frac{\pi}{4}$.

Q06. Evaluate : $\tan\left[2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right]$.

Sol. $\tan\left[2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right] = \frac{\tan\left(2\tan^{-1}\frac{1}{5}\right) - \tan\frac{\pi}{4}}{1 + \tan\left(2\tan^{-1}\frac{1}{5}\right)\tan\frac{\pi}{4}}$

$$\Rightarrow \quad = \frac{\tan\left(\tan^{-1}\dfrac{\frac{2}{5}}{1-\frac{1}{25}}\right)-1}{1+\tan\left(\tan^{-1}\dfrac{\frac{2}{5}}{1-\frac{1}{25}}\right).1} = \frac{\tan\left(\tan^{-1}\frac{5}{12}\right)-1}{1+\tan\left(\tan^{-1}\frac{5}{12}\right)} = \frac{\frac{5}{12}-1}{1+\frac{5}{12}} = -\frac{7}{17}.$$

Q07. Find the inverse (A^{-1}) of the matrix $A=\begin{pmatrix}-1 & 4\\ 7 & 20\end{pmatrix}$, using elementary operations.

Sol. Using elementary row operations, we have : $A = IA$

$$\therefore \begin{pmatrix}-1 & 4\\ 7 & 20\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}A$$

By $R_1 \to (-1)R_1$, $\begin{pmatrix}1 & -4\\ 7 & 20\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}A$

By $R_2 \to R_2 - 7R_1$, $\begin{pmatrix}1 & -4\\ 0 & 48\end{pmatrix} = \begin{pmatrix}-1 & 0\\ 7 & 1\end{pmatrix}A$

By $R_1 \to R_1 + \frac{1}{12}R_2$, $\begin{pmatrix}1 & 0\\ 0 & 48\end{pmatrix} = \begin{pmatrix}-\frac{5}{12} & \frac{1}{12}\\ 7 & 1\end{pmatrix}A$

By $R_2 \to \frac{1}{48}R_2$, $\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \begin{pmatrix}-\frac{5}{12} & \frac{1}{12}\\ \frac{7}{48} & \frac{1}{48}\end{pmatrix}A$

As $I = A^{-1}A$ $\quad \therefore A^{-1} = \begin{pmatrix}-\frac{5}{12} & \frac{1}{12}\\ \frac{7}{48} & \frac{1}{48}\end{pmatrix} = \frac{1}{48}\begin{pmatrix}-20 & 4\\ 7 & 1\end{pmatrix}$.

Q08. If the curves $y = 2e^x$ and $y = ae^{-x}$ intersect orthogonally, find the value of a.

Sol. We have $y = 2e^x$ and $y = ae^{-x}$ $\quad \therefore \frac{dy}{dx} = 2e^x$ and $\frac{dy}{dx} = -ae^{-x}$.

As these curves cut orthogonally, so $(2e^x)\times(-ae^{-x}) = -1 \quad \therefore a = 1/2$.

Q09. Find the points on the curve $y = 12x - x^3$, where the tangent drawn is parallel to x-axis.

Sol. Given curves is $y = 12x - x^3$

Let the point be P(a, b). Clearly we have $b = 12a - a^3$...(i)

Now $\frac{dy}{dx} = 12 - 3x^2 \Rightarrow \left.\frac{dy}{dx}\right]_{\text{at P}} = 12 - 3a^2 = 0$ (As any line parallel to x-axis has slope as 0.)

$\therefore a = 2, -2$. So, by (i), $b = 16, -16$.

Hence the required points are (2, 16) and (–2, –16).

Q10. Find : $\int \sqrt{x^2 - 4x + 13}\,dx$.

Sol. $\int \sqrt{x^2 - 4x + 13}\,dx = \int \sqrt{(x-2)^2 + 3^2}\,dx = \frac{x-2}{2}\sqrt{x^2 - 4x + 13} + \frac{9}{2}\log\left|x - 2 + \sqrt{x^2 - 4x + 13}\right| + c$.

We've used : $\int\sqrt{x^2+a^2}dx=\frac{x}{2}\sqrt{x^2+a^2}+\frac{a^2}{2}\log\left|x+\sqrt{x^2+a^2}\right|+c$ and,

$\sqrt{(x-2)^2+3^2}=\sqrt{x^2-4x+13}$.

Q11. Find a vector perpendicular to the plane of ABC, where A, B and C are points (3, –1, 2), (1, –1, –3) and (4, –3, 1) respectively.

Sol. Let $\vec{a}=\overrightarrow{AB}=(\hat{i}-\hat{j}-3\hat{k})-(3\hat{i}-\hat{j}+2\hat{k})=-2\hat{i}-5\hat{k}$ and,

$\vec{b}=\overrightarrow{AC}=(4\hat{i}-3\hat{j}+\hat{k})-(3\hat{i}-\hat{j}+2\hat{k})=\hat{i}-2\hat{j}-\hat{k}$.

$$\therefore \vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -2 & 0 & -5\\ 1 & -2 & -1\end{vmatrix}=-10\hat{i}-7\hat{j}+4\hat{k}$$

Hence required vector $=\pm\frac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}=\pm\frac{-10\hat{i}-7\hat{j}+4\hat{k}}{\sqrt{100+49+16}}=\mp\frac{10\hat{i}+7\hat{j}-4\hat{k}}{\sqrt{165}}$.

Q12. Eight cards numbered 1 to 8 (one number on one card) are placed in a box, mixed up thoroughly and then a card is drawn randomly. If it is known that the number on the drawn card is more than 2, then find the probability that it is an odd number.

Sol. Let E : the number on card is odd, F : the number on the card is more than 2.

$\therefore S=\{1,2,...,8\}$, $E=\{1,3,5,7\}$, $F=\{3,4,5,6,7,8\}$ $\therefore E\cap F=\{3,5,7\}$

Therefore, $P(E\,|\,F)=\frac{P(E\cap F)}{P(F)}=\frac{\frac{3}{8}}{\frac{6}{8}}=\frac{3}{6}$ $\quad\therefore P(E\,|\,F)=\frac{1}{2}$.

SECTION – C

Q13. Using properties of determinants, prove that : $\begin{vmatrix}a & b-c & b+c\\ a+c & b & c-a\\ a-b & b+a & c\end{vmatrix}=(a+b+c)(a^2+b^2+c^2)$.

OR If $A.\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\end{pmatrix}=\begin{pmatrix}-7 & -8 & -9\\ 2 & 4 & 6\end{pmatrix}$, find the matrix A.

Sol. **LHS :** Let $\Delta=\begin{vmatrix}a & b-c & b+c\\ a+c & b & c-a\\ a-b & b+a & c\end{vmatrix}$ Applying $C_1\to aC_1$

$\Rightarrow\Delta=\frac{1}{a}\begin{vmatrix}a^2 & b-c & b+c\\ a^2+ac & b & c-a\\ a^2-ab & b+a & c\end{vmatrix}$ Applying $C_1\to C_1+bC_2+cC_3$

$\Rightarrow\Delta=\frac{1}{a}\begin{vmatrix}a^2+b^2+c^2 & b-c & b+c\\ a^2+b^2+c^2 & b & c-a\\ a^2+b^2+c^2 & b+a & c\end{vmatrix}$ Taking $(a^2+b^2+c^2)$ common from C_1,

$\Rightarrow\Delta=\frac{a^2+b^2+c^2}{a}\begin{vmatrix}1 & b-c & b+c\\ 1 & b & c-a\\ 1 & b+a & c\end{vmatrix}$ By $R_1\to R_1-R_2$, $R_2\to R_2-R_3$

$\Rightarrow \Delta = \dfrac{a^2+b^2+c^2}{a}\begin{vmatrix} 0 & -c & b+a \\ 0 & -a & -a \\ 1 & b+a & c \end{vmatrix}$ Expanding along C_1,

$\Rightarrow \Delta = \dfrac{a^2+b^2+c^2}{a}\left[0-0+1\left(ac+ab+a^2\right)\right]$

$\therefore \Delta = (a+b+a)(a^2+b^2+c^2) =$ **RHS**.

OR Let $A = \begin{pmatrix} a & x \\ c & d \end{pmatrix}$.

As $A\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$ so, $\begin{pmatrix} a & x \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a+4x & 2a+5x & 3a+6x \\ c+4d & 2c+5d & 3c+6d \end{pmatrix} = \begin{pmatrix} -7 & -8 & -9 \\ 2 & 4 & 6 \end{pmatrix}$

By equality of matrices, we get :

$a+4x=-7,\ 2a+5x=-8,\ 3a+6x=-9,\ c+4d=2,\ 2c+5d=4,\ 3c+6d=6$.

On solving these equations, we get : $a=1,\ x=-2,\ c=2,\ d=0$.

Hence $A = \begin{pmatrix} 1 & -2 \\ 2 & 0 \end{pmatrix}$.

Q14. Find the value of k for which the given function f (x) is continuous at $x=0$,

$$f(x) = \begin{cases} \dfrac{2-2\cos 2x}{x^2};\ x<0 \\ k;\ x=0 \\ \dfrac{\sqrt{x}}{\sqrt{4+\sqrt{x}}-2};\ x>0 \end{cases}.$$

OR If $x^y = e^{x-y}$, show that $\dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

Sol. Here $f(0)=k$...(i)

LHL (at $x=0$): $\lim\limits_{x\to 0^-}\dfrac{2-2\cos 2x}{x^2} = \lim\limits_{x\to 0^-}\dfrac{2(1-\cos 2x)}{x^2} = \lim\limits_{x\to 0^-}\dfrac{2\times 2\sin^2 x}{x^2} = 4\times(1)^2 = 4$ ...(ii)

RHL (at $x=0$): $\lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{\sqrt{4+\sqrt{x}}-2} = \lim\limits_{x\to 0^+}\dfrac{\sqrt{x}}{\sqrt{4+\sqrt{x}}-2}\times\dfrac{\sqrt{4+\sqrt{x}}+2}{\sqrt{4+\sqrt{x}}+2} = \lim\limits_{x\to 0^+}\dfrac{\sqrt{x}\times(\sqrt{4+\sqrt{x}}+2)}{4+\sqrt{x}-4}$

$\Rightarrow \quad = \lim\limits_{x\to 0^+}(\sqrt{4+\sqrt{x}}+2) = \sqrt{4+0}+2 = 4$ ...(iii)

As f (x) is continuous at $x=0$ so, $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^-} f(x) = f(0)$

$\therefore$ By (i), (ii) and (iii), we get : $k=4$.

OR We have $x^y = e^{x-y}$ (Taking logarithm on both sides)

$\Rightarrow \log x^y = \log e^{x-y}$ $\Rightarrow y\log x = (x-y)\log e = x-y$ $\Rightarrow y(1+\log x) = x$ ($\because \log e = 1$

Now, $y = \dfrac{x}{1+\log x}$ $\Rightarrow \dfrac{dy}{dx} = \dfrac{(1+\log x)\times\dfrac{d}{dx}(x) - x\times\dfrac{d}{dx}(1+\log x)}{(1+\log x)^2}$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{(1+\log x)\times 1 - x\times\dfrac{1}{x}}{(1+\log x)^2}$ $\therefore \dfrac{dy}{dx} = \dfrac{\log x}{(1+\log x)^2}$.

Q15. If $x = a(\cos t + t\sin t)$ and $y = a(\sin t - t\cos t)$, find $\dfrac{d^2y}{dx^2}$.

Sol. Given $x = a\left(\cos t + t\sin t\right)$, $y = a\left(\sin t - t\cos t\right)$

Diff. w.r.t. t, we get : $\dfrac{dx}{dt} = a\left(-\sin t + t\cos t + \sin t.1\right)$, $\dfrac{dy}{dt} = a\left(\cos t + t\sin t - \cos t.1\right)$

$\Rightarrow \dfrac{dx}{dt} = at\cos t$, $\dfrac{dy}{dt} = at\sin t$ $\quad \because \dfrac{dy}{dx} = \left(\dfrac{dy}{dt}\right)\left(\dfrac{dt}{dx}\right)$

$\therefore \dfrac{dy}{dx} = \left(at\sin t\right)\left(\dfrac{1}{at\cos t}\right) = \tan t$ $\quad \Rightarrow \dfrac{d^2y}{dx^2} = \sec^2 t \times \dfrac{dt}{dx}$ $\quad \Rightarrow \dfrac{d^2y}{dx^2} = \sec^2 t \times \dfrac{1}{at\cos t}$

Hence, $\dfrac{d^2y}{dx^2} = \dfrac{\sec^3 t}{a\,t}$.

Q16. Find : $\displaystyle\int \frac{x^2}{(x^2+1)(x^2+4)}dx$.

OR Find : $\displaystyle\int \frac{(x-5)e^x}{(x-3)^3}dx$.

Sol. Let $I = \displaystyle\int \frac{x^2}{(x^2+1)(x^2+4)}dx$

Consider $\dfrac{y}{(y+1)(y+4)} = \dfrac{A}{y+1} + \dfrac{B}{y+4}$, where $y = x^2$

$\Rightarrow y = A(y+4) + B(y+1)$ $\quad \therefore A = -\dfrac{1}{3},\ B = \dfrac{4}{3}$ (On comparing the coefficients of the like terms)

So, $I = -\dfrac{1}{3}\displaystyle\int \frac{dx}{x^2+1} + \frac{4}{3}\int \frac{dx}{x^2+4}$ $\quad \Rightarrow I = -\dfrac{1}{3}\tan^{-1}x + \dfrac{4}{3} \times \dfrac{1}{2}\tan^{-1}\dfrac{x}{2} + c$

Hence, $I = \dfrac{2}{3}\tan^{-1}\dfrac{x}{2} + -\dfrac{1}{3}\tan^{-1}x + c$.

OR Let $I = \displaystyle\int \frac{(x-5)e^x}{(x-3)^3}dx$ $\quad \Rightarrow I = \displaystyle\int\left[\frac{(x-3)-2}{(x-3)^3}\right]e^x dx$

$\Rightarrow I = \displaystyle\int\left[\frac{1}{(x-3)^2} + \frac{-2}{(x-3)^3}\right]e^x dx$ $\quad \Rightarrow I = \displaystyle\int \frac{1}{(x-3)^2} \times e^x dx - \int\left[\frac{2}{(x-3)^3}\right]e^x dx$

$\Rightarrow I = \dfrac{1}{(x-3)^2}\displaystyle\int e^x dx - \int\left(\frac{d}{dx}\left(\frac{1}{(x-3)^2}\right)\int e^x dx\right)dx - \int\left[\frac{2}{(x-3)^3}\right]e^x dx$

$\Rightarrow I = \dfrac{1}{(x-3)^2} \times e^x + \displaystyle\int\left[\frac{2}{(x-3)^3}\right]e^x dx - \int\left[\frac{2}{(x-3)^3}\right]e^x dx = \frac{1}{(x-3)^2} \times e^x + C$

$\therefore I = \dfrac{e^x}{(x-3)^2} + C$.

Q17. Evaluate : $\displaystyle\int_0^{\pi} \frac{x\sin x}{4+\cos^2 x}dx$

Sol. Let $I = \displaystyle\int_0^{\pi} \frac{x\sin x}{4+\cos^2 x}dx$...(i) $\quad \Rightarrow I = \displaystyle\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{4+\cos^2(\pi-x)}dx = \int_0^{\pi} \frac{(\pi-x)\sin x}{4+\cos^2 x}dx$...(ii)

Adding (i) and (ii), we get : $2I = \pi\displaystyle\int_0^{\pi} \frac{\sin x}{4+\cos^2 x}dx$

Let $f(x)=\dfrac{\sin x}{4+\cos^2 x}=f(\pi-x)$, so by using $\int_0^{2a} f(x)dx = 2\int_0^{a} f(x)dx$ if $f(2a-x)=f(x)$,

We get : $2I=2\pi\int_0^{\pi/2}\dfrac{\sin x}{4+\cos^2 x}dx$ $\left[\begin{array}{l}\text{Put } \cos x=t \Rightarrow \sin x dx=-dt\\ \text{When } x=0,\ t=1, \text{ when } x=\pi/2,\ t=0\end{array}\right.$

$\therefore I=\pi\int_1^0\dfrac{-dt}{4+t^2}=\pi\int_0^1\dfrac{dt}{4+t^2}=\pi\times\dfrac{1}{2}\left[\tan^{-1}\dfrac{t}{2}\right]_0^1=\dfrac{\pi}{2}\left(\tan^{-1}\dfrac{1}{2}-0\right)$

Hence, $I=\dfrac{\pi}{2}\cdot\tan^{-1}\left(\dfrac{1}{2}\right)$.

Q18. Find the general solution of the following differential equation :

$$ydx+x\log\left(\frac{y}{x}\right)dy-2xdy=0.$$

Sol. Here $ydx+x\log\left(\dfrac{y}{x}\right)dy-2xdy=0 \Rightarrow \dfrac{dy}{dx}=\dfrac{y}{2x-x\log\left(\dfrac{y}{x}\right)}=\dfrac{\dfrac{y}{x}}{2-\log\left(\dfrac{y}{x}\right)}$

Put $y=vx \Rightarrow \dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

So, $v+x\dfrac{dv}{dx}=\dfrac{v}{2-\log v}$ $\Rightarrow x\dfrac{dv}{dx}=\dfrac{v}{2-\log v}-v=\dfrac{v-2v+v\log v}{2-\log v}=\dfrac{v\log v-v}{2-\log v}$

$\Rightarrow\int\dfrac{2-\log v}{v(\log v-1)}dv=\int\dfrac{dx}{x}$ (In the integral of LHS, put $\log v=t\Rightarrow\dfrac{dv}{v}=dt$.

So, $\int\dfrac{2-t}{t-1}dt=\int\dfrac{dx}{x}$ $\Rightarrow\int\left(-1+\dfrac{1}{t-1}\right)dt=\int\dfrac{dx}{x}$ $\Rightarrow\log|t-1|-t=\log|x|+c$

$\therefore\log\left|\log\left(\dfrac{y}{x}\right)-1\right|-\log\left(\dfrac{y}{x}\right)=\log|x|+c$ $\Rightarrow\log\left|\log\left(\dfrac{y}{x}\right)-1\right|=\log|x|+\log\left(\dfrac{y}{x}\right)+c$

$\Rightarrow\log\left|\log\left(\dfrac{y}{x}\right)-1\right|=\log\left(x\times\dfrac{y}{x}\right)+c$ $\Rightarrow\log\left|\log\left(\dfrac{y}{x}\right)-1\right|=\log y+c$

Hence, $\log\left|\log\left(\dfrac{y}{x}\right)-1\right|=\log y+c$ is the required general solution.

Q19. For the differential equation $xy\dfrac{dy}{dx}=(x+2)(y+2)$, find the solution curve passing through the point (1, –1).

Sol. We have $xy\dfrac{dy}{dx}=(x+2)(y+2)$ $\Rightarrow\dfrac{ydy}{y+2}=\dfrac{x+2}{x}dx$ $\Rightarrow\int\left(1-\dfrac{2}{y+2}\right)dy=\int\left(1+\dfrac{2}{x}\right)dx$

$\Rightarrow y-2\log|y+2|=x+2\log|x|+c \ldots$(i)

As (i) passes through (1, –1) so, $-1-2\log|-1+2|=1+2\log|1|+c$ i.e., $c=-2$.

Hence the required curve is $y-2\log|y+2|=x+2\log|x|-2$.

Q20. If the scalar product of $\hat{i}+\hat{j}+\hat{k}$ with a unit vector along the sum of vectors $2\hat{i}+4\hat{j}-5\hat{k}$ and $\lambda\hat{i}+2\hat{j}+3\hat{k}$ is equal to 1, find the value of λ.

Sol. Let $\vec{a}=2\hat{i}+4\hat{j}-5\hat{k}+\lambda\hat{i}+2\hat{j}+3\hat{k}=(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}$

$\therefore \hat{a} = \dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+6^2+(-2)^2}} = \dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}}$

According to the conditions, $(\hat{i}+\hat{j}+\hat{k}).\hat{a} = (\hat{i}+\hat{j}+\hat{k}).\dfrac{(2+\lambda)\hat{i}+6\hat{j}-2\hat{k}}{\sqrt{(2+\lambda)^2+40}} = 1$

$\Rightarrow (2+\lambda)+6-2 = \sqrt{(2+\lambda)^2+40}$ $\qquad \Rightarrow (6+\lambda)^2 = (2+\lambda)^2+40$

$\Rightarrow 36+\lambda^2+12\lambda = 4+\lambda^2+4\lambda+40$ $\qquad \Rightarrow 8\lambda = 8$ $\qquad \therefore \lambda = 1$.

Q21. Find the coordinates of the foot of perpendicular drawn from the point P(1, 8, 4) to the line joining the points A(0, –1, 3) and B(2, –3, –1). Also find the length of this perpendicular.

Sol. Equation of line AB : $\dfrac{x-0}{2-0} = \dfrac{y-(-1)}{-3-(-1)} = \dfrac{z-3}{-1-3}$ i.e., $\dfrac{x}{2} = \dfrac{y+1}{-2} = \dfrac{z-3}{-4} = \lambda$ *say*

$\therefore$ Coordinates of random point on this line is $Q(2\lambda, -2\lambda-1, -4\lambda+3)$.

The d.r.'s of PQ are $2\lambda-1, -2\lambda-9, -4\lambda-1$.

As $PQ \perp AB$ so, $2(2\lambda-1)-2(-2\lambda-9)-4(-4\lambda-1)=0$

$\Rightarrow \lambda = -5/6$

$\therefore Q\left(-\dfrac{5}{3}, \dfrac{2}{3}, \dfrac{19}{3}\right)$ is the foot of perpendicular.

Also, length of perpendicular is, $PQ = \sqrt{\left(-\dfrac{5}{3}-1\right)^2+\left(\dfrac{2}{3}-8\right)^2+\left(\dfrac{19}{3}-4\right)^2} = \dfrac{\sqrt{597}}{3}$ units.

Q22. Find the probability distribution of number of doublets in two throws of a pair of dice. Hence find the mean of the distribution.

Sol. Let X : number of doublets in two throws of a pair of dice. $\therefore X = 0,1,2$.

Here n = 2. Let p be the probability of success $\therefore p = \dfrac{6}{36} = \dfrac{1}{6}$, $q = 1-p = \dfrac{5}{6}$

Therefore by using, $P(X=r) = {}^nC_r p^r q^{n-r}$ we have : $P(X=r) = {}^2C_r \times \dfrac{5^{2-r}}{36}$

X	P(X)	X P(X)
0	$\dfrac{25}{36}$	0
1	$\dfrac{10}{36}$	$\dfrac{10}{36}$
2	$\dfrac{1}{36}$	$\dfrac{2}{36}$
		$\sum X\ P(X) = \dfrac{12}{36}$

Therefore, mean $= \sum X\,P(X) = \dfrac{1}{3}$.

Q23. A and B throw a pair of dice alternatively till one of them gets a sum of 5, of the numbers on the two dice and wins the game. Find their respective probabilities of winning, if A starts the game.

Sol. Sum of 5 is obtained when we get following outcomes on the throw of pair of dice :

(1, 4), (4, 1), (2, 3), (3, 2)

Let E : getting the sum of 5. So, $P(E) = \dfrac{4}{36} = \dfrac{1}{9}$ $\quad \therefore P(\bar{E}) = 1-P(E) = \dfrac{8}{9}$.

If A starts the game then, he can win in first, third, fifth, … throws.

So, $P(\text{A wins}) = P(E) + P(\overline{E}\,\overline{E}E) + P(\overline{E}\,\overline{E}\,\overline{E}\,\overline{E}E) + ... = \frac{1}{9} + \left(\frac{8}{9}\right)^2 \frac{1}{9} + \left(\frac{8}{9}\right)^4 \frac{1}{9} + ...$

$\Rightarrow P(\text{A wins}) = \dfrac{\frac{1}{9}}{1-\frac{64}{81}} = \frac{9}{17}$ $\qquad [\because a + ar + ar^2 + ...\infty = \frac{a}{1-r}$

And, $P(\text{B wins}) = 1 - P(\text{A wins}) = 1 - \frac{9}{17} = \frac{8}{17}$.

SECTION – D

Q24. Let R be the relation defined in the set $A = \{1,2,3,4,5,6,7,8,9\}$ by $R = \{(x, y) : x, y \in A$, x and y are either both odd or both even}. Show that R is an equivalence relation. Write all the equivalence classes of set A.

OR Let A be the set of all real numbers except –1 and let * be a binary operation on A defined by a * b = a + b + ab, $\forall\ a, b \in A$. Prove that (i) * is commutative and associative, and (ii) number 0 is its identity element.

Sol. We've $R = \{(x, y) : x, y \in A$, x and y are either both odd or both even} and,

$A = \{1,2,3,4,5,6,7,8,9\}$.

Reflexivity : Let any element a ∈ A, both a and a must be either odd or even, so that (a, a) ∈ R.

Symmetry : Let (a, b) ∈ R ⇒ both a and b must be either odd or even ⇒ (b, a) ∈ R. So, R is symmetric.

Transitivity : Let (a, b) ∈ R and (b, c) ∈ R ⇒ all elements a, b, c, must be either even or odd simultaneously ⇒ (a, c) ∈ R. Hence, R is a transitive relation.

Since the relation R is reflexive, symmetric and transitive so, it is an equivalence relation.

Now let (1, x) $\in R$, clearly x will be odd. Hence [1] = {1, 3, 5, 7, 9}.

Similarly [3] = [5] = [7] = [9] = {1, 3, 5, 7, 9}.

Also let (2, y) $\in R$, clearly y will be even. Hence [2] = {2, 4, 6, 8}.

Similarly [2] = [4] = [6] = [8] = {2, 4, 6, 8}.

OR We've a * b = a + b + ab, $\forall\ a, b \in A$, where A is the set of real numbers except –1.

(i) Commutativity : Let $a, b \in A$. As $a * b = a + b + ab = b + a + ba = b * a$, so * is commutative.

Associativity : Let Let $a, b, c \in A$.

Now $(a*b)*c = (a+b+ab)*c = a+b+c+ab+bc+ca+abc$ …(A)

Also, $a*(b*c) = a*(b+c+bc) = a+b+c+ab+bc+ca+abc$ …(B)

By (A) and (B), $(a*b)*c = a*(b*c)$. Hence * is associative.

(ii) Let e be the identity element.

Then, $a*e = a = e*a$ for all $a \in A = R - \{-1\}$ $\qquad \Rightarrow a*e = a$ and $e*a = a$

$\Rightarrow a + e + ae = a$ and $e + a + ea = a$ $\qquad \Rightarrow e(1+a) = 0$ and $e(1+a) = 0$

$\therefore e = 0$ $\qquad [\because\ a \in R - \{-1\} \Rightarrow a \neq -1 \Rightarrow 1 + a \neq 0]$

Hence 0 is the identity element for * defined in set A = R–{–1}.

Q25. Solve the following system of equations, using matrix method :

$5x - y + z = 4,\ 3x + 2y - 5z = 2,\ x + 3y - 2z = 5$.

Sol. We have $5x - y + z = 4,\ 3x + 2y - 5z = 2,\ x + 3y - 2z = 5$

Let $A = \begin{bmatrix} 5 & -1 & 1 \\ 3 & 2 & -5 \\ 1 & 3 & -2 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $B = \begin{bmatrix} 4 \\ 2 \\ 5 \end{bmatrix}$.

As $AX = B$ $\qquad \Rightarrow X = A^{-1}B$...(i)

Here $A=\begin{bmatrix}5 & -1 & 1\\ 3 & 2 & -5\\ 1 & 3 & -2\end{bmatrix}$ $\therefore |A|=\begin{vmatrix}5 & -1 & 1\\ 3 & 2 & -5\\ 1 & 3 & -2\end{vmatrix}=55-1+7=61\neq 0 \therefore A^{-1}$ exists.

Consider C_{ij} be the cofactor of corresponding element a_{ij} of matrix A.

$C_{11}=11, C_{12}=1, C_{13}=7,$
$C_{21}=1, C_{22}=-11, C_{23}=-16,$
$C_{31}=3, C_{32}=28, C_{33}=13$

$\therefore adj.A=\begin{bmatrix}11 & 1 & 3\\ 1 & -11 & 28\\ 7 & -16 & 13\end{bmatrix}$

$$\Rightarrow A^{-1}=\frac{adj.A}{|A|}=\frac{1}{61}\begin{bmatrix}11 & 1 & 3\\ 1 & -11 & 28\\ 7 & -16 & 13\end{bmatrix}$$

By (i), we get : $X=\frac{1}{61}\begin{bmatrix}11 & 1 & 3\\ 1 & -11 & 28\\ 7 & -16 & 13\end{bmatrix}\begin{bmatrix}4\\2\\5\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\\z\end{bmatrix}=\frac{1}{61}\begin{bmatrix}44+2+15\\ 4-22+140\\ 28-32+65\end{bmatrix}=\begin{bmatrix}1\\2\\1\end{bmatrix}$

By equality of matrices, we get : $x=1, y=2, z=1$.

Q26. Find the intervals in which the function f given by $f(x)=\sin 3x-\cos 3x, 0<x<\pi$, is (a) strictly increasing and, (b) strictly decreasing.

OR Prove that the height of a solid cylinder of given surface and greatest volume is equal to the diameter of its base.

Sol. Given function is $f(x)=\sin 3x-\cos 3x, 0<x<\pi$

$\Rightarrow f'(x)=3\cos 3x+3\sin 3x=3(\cos 3x+\sin 3x)$

For critical points, $f'(x)=3(\cos 3x+\sin 3x)=0 \quad \Rightarrow \tan 3x=-1$

$$\Rightarrow 3x=\frac{3\pi}{4},\frac{7\pi}{4},\frac{11\pi}{4},\frac{15\pi}{4},...$$

$$\therefore x=\frac{\pi}{4},\frac{7\pi}{12},\frac{11\pi}{12}\in(0,\pi).$$

Interval	Sign of $f'(x)$	$f(x)$ is strictly
$(0,\pi/4)$	Positive	Increasing
$(\pi/4,7\pi/12)$	Negative	Decreasing
$(7\pi/12,11\pi/12)$	Positive	Increasing
$(11\pi/12,\pi)$	Negative	Decreasing

Hence, f (x) is (a) strictly increasing in the interval $\left(0,\frac{\pi}{4}\right)$ and $\left(\frac{7\pi}{12},\frac{11\pi}{12}\right)$ and

(b) strictly decreasing in the interval $\left(\frac{\pi}{4},\frac{7\pi}{12}\right)$ and $\left(\frac{11\pi}{12},\pi\right)$.

OR Given surface area, $S=2\pi rh+2\pi r^2 \Rightarrow h=\frac{S-2\pi r^2}{2\pi r}$ ...(i)

Now volume of cylinder, $V=\pi r^2 h \quad \Rightarrow V=\pi r^2\times\frac{S-2\pi r^2}{2\pi r}=\frac{Sr-2\pi r^3}{2}$

$$\therefore \frac{dV}{dr}=\frac{S-6\pi r^2}{2} \text{ and, } \frac{d^2V}{dr^2}=-6\pi r$$

For local points of maxima and/or minima, $\frac{dV}{dr}=\frac{S-6\pi r^2}{2}=0 \quad \Rightarrow r=\sqrt{\frac{S}{6\pi}}$

$\because \left.\dfrac{d^2V}{dr^2}\right]_{\text{at } r=\sqrt{\frac{S}{6\pi}}} = -6\pi\left(\sqrt{\dfrac{S}{6\pi}}\right) < 0$ $\therefore$ V is maximum at $r = \sqrt{\dfrac{S}{6\pi}}$.

Now we know that $S = 6\pi r^2$ (from above steps)

By (i), $2\pi r h + 2\pi r^2 = 6\pi r^2$ $\Rightarrow 2\pi r h = 4\pi r^2$ $\Rightarrow 2h = 4r$ $\therefore h = 2r$

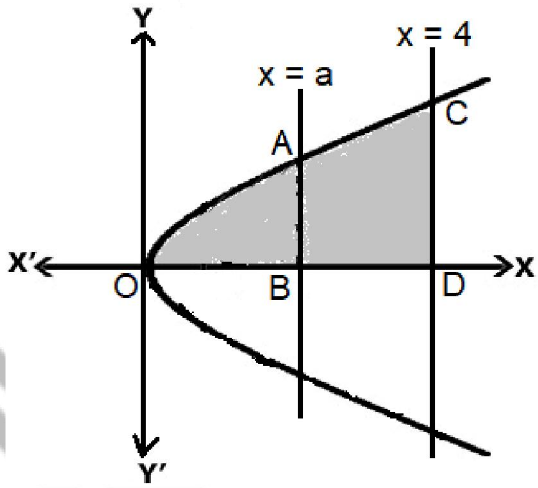
Therefore, the height of a solid cylinder of given surface area and greatest volume is equal to the diameter of its base.

Q27. If the area between the curve $x = y^2$ and the line $x = 4$ is divided into two equal parts by the line $x = a$, find the value of a.

OR

By the method of limit of sum, find the value of the following definite integral :

$$\int_1^3 (3x^2 + 2x + e^x)dx$$



Sol. We have $x = y^2$, $x = 4$ and $x = a$.

According to question,

ar(OABO) = ar(ACDBA)

$\Rightarrow \int_0^a \sqrt{x}dx = \int_a^4 \sqrt{x}dx$

$\Rightarrow \dfrac{2}{3}\left[x^{3/2}\right]_o^a = \dfrac{2}{3}\left[x^{3/2}\right]_a^4$

$\Rightarrow a^{3/2} - 0 = 4^{3/2} - a^{3/2}$

$\Rightarrow 2a^{3/2} = (4^{1/2})^3$

$\Rightarrow 2a^{3/2} = 8$

$\Rightarrow a^{3/2} = 4$ $\Rightarrow a^3 = 16$ $\therefore a = \sqrt[3]{16}$ or, $16^{1/3}$.

OR Let $I = \int_1^3 (3x^2 + 2x + e^x)dx$

We know $\int_a^b f(x)dx = \lim_{n\to\infty} h\left[f(a) + f(a+h) + f(a+2h) + ... + f(a+(n-1)h)\right]$,

As $n \to \infty, h \to 0 \Rightarrow nh = b - a$

$\therefore \int_a^b f(x)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1} f(a+rh)$ ...(i)

Here $f(x) = 3x^2 + 2x + e^x$, $a = 1$, $b = 3$ $\therefore f(a+rh) = 3(a+rh)^2 + 2(a+rh) + e^{a+rh}$

$\Rightarrow f(1+rh) = 3(1+rh)^2 + 2(1+rh) + e^{1+rh} = 5 + 8rh + 3r^2h^2 + ee^{rh}$.

By using (i), $\int_1^3 \left(3x^2 + 2x + e^x\right)dx = \lim_{n\to\infty} h\sum_{r=0}^{n-1}\left[5 + 8rh + 3r^2h^2 + ee^{rh}\right]$

$\Rightarrow I = \lim_{n\to\infty} h\left\{3h^2\sum_{r=0}^{n-1} r^2 + 8h\sum_{r=0}^{n-1} r + \sum_{r=0}^{n-1} 5 + e\sum_{r=0}^{n-1} e^{rh}\right\}$

$\Rightarrow I = \lim_{n\to\infty} h\left\{3h^2\dfrac{n(n-1)(2n-1)}{6} + 8h\dfrac{n(n-1)}{2} + 5n + e\left(1 + e^h + e^{2h} + ... + e^{(n-1)h}\right)\right\}$

$\Rightarrow I = \lim_{n\to\infty}\left\{\dfrac{nh(nh-h)(2nh-h)}{2} + 4 \times nh(nh-h) + 5nh + e \times (e^{nh} - 1)\left(\dfrac{h}{e^h - 1}\right)\right\}$

Since $n \to \infty, h \to 0 \Rightarrow nh = 3 - 1 = 2$.

So, $I=\frac{2(2-0)(4-0)}{2}+4\times 2\times(2-0)+5\times 2+e\times(e^2-1)(1)=34+(e^3-e)$

Hence, $I=\int_1^3 (3x^2+2x+e^x)dx=34+e^3-e$.

Q28. Show that the lines $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z}{-1}$ and $\frac{x-4}{3}=\frac{1-y}{2}=z-1$ are coplanar.
Hence find the equation of the plane containing these lines.

Sol. The lines are $\frac{x-1}{2}=\frac{y-3}{4}=\frac{z}{-1}$ and $\frac{x-4}{3}=\frac{1-y}{2}=z-1$

Rewriting these lines in vector form, we get :

$\vec{r}=\hat{i}+3\hat{j}+\lambda(2\hat{i}+4\hat{j}-\hat{k})$ and $\vec{r}=4\hat{i}+\hat{j}+\hat{k}+\lambda(3\hat{i}-2\hat{j}+\hat{k})$

Here $\vec{a}_1=\hat{i}+3\hat{j}, \vec{b}_1=2\hat{i}+4\hat{j}-\hat{k}$, $\vec{a}_2=4\hat{i}+\hat{j}+\hat{k}, \vec{b}_2=3\hat{i}-2\hat{j}+\hat{k}$.

As $\vec{a}_2-\vec{a}_1=3\hat{i}-2\hat{j}+\hat{k}$.

Now $(\vec{a}_2-\vec{a}_1).(\vec{b}_1\times\vec{b}_2)=\begin{vmatrix}3 & -2 & 1\\ 2 & 4 & -1\\ 3 & -2 & 1\end{vmatrix}=3(2)+2(5)+1(-16)=16-16=0$

$\therefore$ Given lines are coplanar.

Perpendicular vector to the plane $\vec{b}_1\times\vec{b}_2=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & 4 & -1\\ 3 & -2 & 1\end{vmatrix}=2\hat{i}-5\hat{j}-16\hat{k}$

$\therefore$ Eq. of plane : $\vec{r}.(2\hat{i}-5\hat{j}-16\hat{k})=(\hat{i}+3\hat{j}).(2\hat{i}-5\hat{j}-16\hat{k})$ i.e., $\vec{r}.(2\hat{i}-5\hat{j}-16\hat{k})=-13$

Therefore, $2x-5y-16z+13=0$ is the required equation of plane.

Q29. If a class XII student aged 17 years, rides his motor cycle at 40 km/hr, the petrol cost is ₹2 per km. If he rides at a speed of 70 km/hr, the petrol cost increases to ₹7 per km. He has ₹100 to spend on petrol and wishes to cover the maximum distance within one hour.
(i) Express the above as as an LPP.
(ii) What are the benefits of driving a vehicle at a slow speed?
(iii) Should a child below 18 years be allowed to drive a motorcycle? Give reasons.

Sol. (i) Let x and y represent the distance travelled by the student at speed of 40 km/hr and 70 km/hr respectively.

To Maximize : Z = x + y (in km).

Subject to constraints : $\frac{x}{40}+\frac{y}{70}\le 1,\ 2x+7y\le 100,\ x\ge 0,\ y\ge 0$.

(ii) It saves petrol and hence saves money too.

(iii) No, because according to the law driving license is issued when a person is above the 18 years of age.