CBSE 2017 COMPARTMENT EXAMINATION DELHI

(Series GBM/1/C Code No. 65/1/1 : Delhi Region)

Max. Marks : 100 **Time Allowed : 3 Hours**

SECTION – A

Q01. Let A and B are matrices of order 3×2 and 2×4 respectively. Write the order of matrix (AB).

Sol. Order of matrix (AB) is 3×4.

Q02. Write the equation of tangent drawn to the curve $y=\sin x$ at the point (0, 0).

Sol. We have $y=\sin x$ $\quad\Rightarrow \frac{dy}{dx}=\cos x$ $\quad\therefore \left.\frac{dy}{dx}\right]_{at\ (0,0)}=m_T=\cos 0=1$.

Equation of tangent at (0, 0) : $y-0=1(x-0)$ i.e., $x-y=0$.

Q03. Find : $\int \frac{1}{x(1+\log x)}dx$.

Sol. Let $I=\int \frac{1}{x(1+\log x)}dx$ $\quad\left[\text{Put } 1+\log x=t\Rightarrow \frac{dx}{x}=dt\right.$

$\therefore I=\int \frac{1}{t}dt=\log|t|+C=\log|1+\log x|+C$.

Q04. Write the angle between the vectors $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$.

Sol. The required angle is 180° (or, π) (As vectors $\vec{a}\times\vec{b}$ and $\vec{b}\times\vec{a}$ are anti-parallel vectors).

SECTION – B

Q05. In the following matrix equation use elementary operation $R_2\to R_2+R_1$ and write the equation thus obtained

$$\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & -1\end{pmatrix}=\begin{pmatrix}8 & -3\\ 9 & -4\end{pmatrix}$$

Sol. $$\begin{pmatrix}2 & 3\\ 1 & 4\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & -1\end{pmatrix}=\begin{pmatrix}8 & -3\\ 9 & -4\end{pmatrix}$$

Applying $R_2\to R_2+R_1$, we get : $\begin{pmatrix}2 & 3\\ 3 & 7\end{pmatrix}\begin{pmatrix}1 & 0\\ 2 & -1\end{pmatrix}=\begin{pmatrix}8 & -3\\ 17 & -7\end{pmatrix}$.

Q06. Find the value of k for which the function $f(x)=\begin{cases}\frac{x^2+3x-10}{x-2}, & x\neq 2\\ k, & x=2\end{cases}$ is continuous at $x=2$.

Sol. As $f(x)$ is continuous at $x=2$ so, $\lim_{x\to 2}f(x)=f(2)$.

$\therefore \lim_{x\to 2}\frac{x^2+3x-10}{x-2}=k$ $\quad\Rightarrow \lim_{x\to 2}\frac{(x-2)(x+5)}{x-2}=k$ $\quad\Rightarrow \lim_{x\to 2}(x+5)=k$

$\Rightarrow (2+5)=k$ $\quad\therefore k=7$.

Q07. The radius r of a right circular cone is decreasing at the rate of 3 cm/minute and the height h is increasing at the rate of 2 cm/minute. When r = 9 cm and h = 6 cm, find the rate of change of its volume.

Sol. Here $\frac{dr}{dt}=-3$ cm/min, $\frac{dh}{dt}=2$ cm/min.

Now, volume of cone is $V=\frac{1}{3}\pi r^2 h$ $\quad\Rightarrow \frac{dV}{dt}=\frac{\pi}{3}\left[r^2\frac{dh}{dt}+2rh\frac{dr}{dt}\right]$

$\therefore \frac{dV}{dt}=\frac{\pi}{3}\left[9^2\times(2)+2\times 9\times 6\times(-3)\right]=-54\pi\ cm^3/min$.

Hence required rate of change of volume is -54π cm^3/min.

Q08. Find $\int \sqrt{x^2-2x}\,dx$.

Sol. $\int \sqrt{x^2-2x}\,dx = \int \sqrt{(x-1)^2-1^2}\,dx = \frac{x-1}{2}\sqrt{x^2-2x} - \frac{1}{2}\log\left|x-1+\sqrt{x^2-2x}\right| + C$.

Q09. Find the differential equation of the family of curves $y^2 = 4ax$.

Sol. We've $y^2 = 4ax \quad \Rightarrow \frac{y^2}{x} = 4a$

On diff. w. r. to x both sides, we get : $\frac{x\times 2yy' - y^2\times 1}{x^2} = 0 \quad \therefore 2x\frac{dy}{dx} - y = 0$ is the req. diff. eq.

Q10. Find the general solution of the differential equation $\frac{dy}{dx} + 2y = e^{3x}$.

Sol. We've $\frac{dy}{dx} + 2y = e^{3x}$

It is of the form $\frac{dy}{dx} + P(x)y = Q(x)$ where $P(x) = 2,\ Q(x) = e^{3x}$.

Now I. F. $= e^{\int 2dx} = e^{2x}$.

So, the required solution is $y(e^{2x}) = \int e^{2x}\times e^{3x}dx + C$ i.e., $ye^{2x} = \frac{1}{5}e^{5x} + C$ or, $y = \frac{1}{5}e^{3x} + Ce^{-2x}$.

Q11. If the points with position vectors $10\hat{i}+3\hat{j},\ 12\hat{i}-5\hat{j}$ and $\lambda\hat{i}+11\hat{j}$ are collinear, find the value of λ.

Sol. Let $A(10\hat{i}+3\hat{j}),\ B(12\hat{i}-5\hat{j})$ and $C(\lambda\hat{i}+11\hat{j})$.

So $\overrightarrow{AB} = 2\hat{i}-8\hat{j},\ \overrightarrow{BC} = (\lambda-12)\hat{i}+16\hat{j}$.

As the points A, B and C are collinear so, $\frac{2}{\lambda-12} = \frac{-8}{16} \quad \therefore \lambda = 8$.

Q12. A firm has to transport at least 1200 packages daily using large vans which carry 200 packages each and small vans which can take 80 packages each. The cost for engaging each large van is ₹400 and each small van is ₹200. Not more than ₹3000 is to be spent daily on the job and the number of large vans cannot exceed the number of small vans. Formulate this problem as a LPP given that the objective is to minimize the cost.

Sol. Let number of large van and small van be x and y respectively.

To minimize : Z = ₹(400x + 200y)

Subject to constraints : $x \ge 0,\ y \ge 0,\ 400x+200y \le 3000,\ 200x+80y \ge 1200,\ x \le y$.

SECTION – C

Q13. Prove that $\tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x,\ -\frac{1}{\sqrt{2}} \le x \le 1$.

Sol. **LHS :** Let $y = \tan^{-1}\left(\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right) = \tan^{-1}\left(\frac{\sqrt{1+\cos 2\theta}-\sqrt{1-\cos 2\theta}}{\sqrt{1+\cos 2\theta}+\sqrt{1-\cos 2\theta}}\right)$ [Put $x = \cos 2\theta$...(i)

$\therefore y = \tan^{-1}\left(\frac{\sqrt{2}|\cos\theta| - \sqrt{2}|\sin\theta|}{\sqrt{2}|\cos\theta| + \sqrt{2}|\sin\theta|}\right) = \tan^{-1}\left(\frac{|\cos\theta|-|\sin\theta|}{|\cos\theta|+|\sin\theta|}\right)$

Note that $-\frac{1}{\sqrt{2}} \le x \le 1 \Rightarrow -\frac{1}{\sqrt{2}} \le \cos 2\theta \le 1 \Rightarrow 0 \le 2\theta \le \frac{3\pi}{4} \Rightarrow 0 \le \theta \le \frac{3\pi}{8}$ i.e., $\theta \in$ I quadrant.

So $y = \tan^{-1}\left(\frac{\cos\theta-\sin\theta}{\cos\theta+\sin\theta}\right) = \tan^{-1}\left(\frac{1-\tan\theta}{1+1.\tan\theta}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}-\theta\right)$

$\Rightarrow y = \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x =$ **RHS**. [By (i).

Q14. If $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$, then using properties of determinants, find the value of $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}$, where $x, y, z \neq 0$.

OR Using elementary operations, find the inverse of the following matrix A

$A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

Sol. Here $\begin{vmatrix} a & b-y & c-z \\ a-x & b & c-z \\ a-x & b-y & c \end{vmatrix} = 0$

Apply $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$, $\begin{vmatrix} x & 0 & -z \\ 0 & y & -z \\ a-x & b-y & c \end{vmatrix} = 0$

On expanding along R_1, we get : $x(cy+bz-yz) - 0 - z(-ay+xy) = 0$

That is, $xcy + bzx - xyz + ayz - xyz = 0 \quad \Rightarrow xcy + bzx + ayz = 2xyz$

Dividing both sides by xyz, we get : $\frac{a}{x}+\frac{b}{y}+\frac{c}{z} = 2$.

OR Here $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Using elementary row operations, $A = IA$ i.e., $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} A$

Applying $R_2 \to R_2 - 2R_1$, $\begin{pmatrix} 1 & 2 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix} A$

Applying $R_2 \to -\frac{1}{5}R_2$, $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2/5 & -1/5 \end{pmatrix} A$

Applying $R_1 \to R_1 - 2R_2$, $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix} A$

As $I = A^{-1}A \quad \therefore A^{-1} = \begin{pmatrix} 1/5 & 2/5 \\ 2/5 & -1/5 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$.

Q15. If $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$, then find $\frac{d^2y}{dx^2}$.

Sol. Given $x = a(\cos\theta + \theta\sin\theta)$ and $y = a(\sin\theta - \theta\cos\theta)$

Diff. w.r.t. θ, we get : $\frac{dx}{d\theta} = a\left(-\sin\theta + \theta\cos\theta + \sin\theta.1\right)$, $\frac{dy}{d\theta} = a\left(\cos\theta + \theta\sin\theta - \cos\theta.1\right)$

$\Rightarrow \frac{dx}{d\theta} = a\theta\cos\theta$, $\frac{dy}{d\theta} = a\theta\sin\theta \qquad \because \frac{dy}{dx} = \left(\frac{dy}{d\theta}\right)\left(\frac{d\theta}{dx}\right)$

$\therefore \frac{dy}{dx} = \left(a\theta\sin\theta\right)\left(\frac{1}{a\theta\cos\theta}\right) = \tan\theta \quad \Rightarrow \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{d\theta}{dx} \quad \Rightarrow \frac{d^2y}{dx^2} = \sec^2\theta \times \frac{1}{a\theta\cos\theta} = \frac{\sec^3\theta}{a\theta}$.

Q16. Find the equation of tangent to the curve $y = \cos(x+y), -2\pi \le x \le 0$, that is parallel to the line $x + 2y = 0$.

Sol. We've $y = \cos(x+y), x \in [-2\pi, 0] \quad \Rightarrow \frac{dy}{dx} = -\sin(x+y)\left(1 + \frac{dy}{dx}\right) \quad \therefore \frac{dy}{dx} = -\frac{\sin(x+y)}{1+\sin(x+y)}$

Slope of line x + 2y = 0 is –1/2, which is also the slope of tangent (as tangent is parallel to line.)

According to the question, $-\dfrac{\sin(x+y)}{1+\sin(x+y)}=-\dfrac{1}{2}$ $\Rightarrow \sin(x+y)=1$ $\Rightarrow \sin(x+y)=\sin\dfrac{\pi}{2}$

$\Rightarrow x+y=n\pi+(-1)^n\dfrac{\pi}{2}, n\in Z$

So, $y=\cos(x+y)=\cos\left[n\pi+(-1)^n\dfrac{\pi}{2}\right], n\in Z$ $\Rightarrow y=0$ for all $n\in Z$.

Also $x\in[-2\pi,0]$ $\Rightarrow x=\dfrac{-3\pi}{2}$. So, the point is $(-3\pi/2,0)$.

Therefore, equation of tangent at $(-3\pi/2,0)$: $y-0=-\dfrac{1}{2}\left(x+\dfrac{3\pi}{2}\right)$ $\Rightarrow 2x+4y+3\pi=0$.

Q17. Evaluate : $\displaystyle\int\frac{x+5}{3x^2+13x-10}dx$. **OR** Evaluate : $\displaystyle\int_0^{\pi/4}\frac{1}{\cos^2 x+4\sin^2 x}dx$.

Sol. Let $I=\displaystyle\int\frac{x+5}{3x^2+13x-10}dx$ $\Rightarrow I=\dfrac{1}{6}\displaystyle\int\frac{6x+30}{3x^2+13x-10}dx$

$\Rightarrow I=\dfrac{1}{6}\displaystyle\int\frac{6x+13+17}{3x^2+13x-10}dx$ $\Rightarrow I=\dfrac{1}{6}\displaystyle\int\frac{6x+13}{3x^2+13x-10}dx+\frac{17}{6}\int\frac{1}{3x^2+13x-10}dx$

$\Rightarrow I=\dfrac{1}{6}\log\left|3x^2+13x-10\right|+\dfrac{17}{6}\times\dfrac{1}{3}\displaystyle\int\frac{1}{x^2+\frac{13}{3}x-\frac{10}{3}}dx$

$\Rightarrow I=\dfrac{1}{6}\log\left|3x^2+13x-10\right|+\dfrac{17}{18}\displaystyle\int\frac{1}{\left(x+\frac{13}{6}\right)^2-\left(\frac{17}{6}\right)^2}dx$

$\Rightarrow I=\dfrac{1}{6}\log\left|3x^2+13x-10\right|+\dfrac{17}{18}\times\dfrac{1}{2\left(\frac{17}{6}\right)}\log\left|\dfrac{x+\frac{13}{6}-\frac{17}{6}}{x+\frac{13}{6}+\frac{17}{6}}\right|+C$

$\Rightarrow I=\dfrac{1}{6}\log\left|3x^2+13x-10\right|+\dfrac{1}{6}\log\left|\dfrac{x-\frac{2}{3}}{x+5}\right|+C$

$\therefore I=\dfrac{1}{6}\log\left|3x^2+13x-10\right|+\dfrac{1}{6}\log\left|\dfrac{3x-2}{x+5}\right|+\lambda$, where $\lambda=C-\dfrac{1}{6}\log 3$.

OR Let $I=\displaystyle\int_0^{\pi/4}\frac{1}{\cos^2 x+4\sin^2 x}dx$

Dividing Nr and Dr both by $\cos^2 x$, we get : $I=\displaystyle\int_0^{\pi/4}\frac{\sec^2 x}{1+4\tan^2 x}dx$

Put $\tan x=t\Rightarrow \sec^2 x dx=dt$.

Also, when $x=0\Rightarrow t=0$ and, when $x=\pi/4\Rightarrow t=1$.

$\therefore I=\displaystyle\int_0^1\frac{dt}{1+4t^2}=\int_0^1\frac{dt}{1+(2t)^2}=\frac{1}{2}\left[\tan^{-1}2t\right]_0^1$

$\Rightarrow I=\dfrac{1}{2}\left[\tan^{-1}2-\tan^{-1}0\right]=\dfrac{1}{2}\tan^{-1}2$.

Q18. Find $\int \frac{x^2}{(x-1)(x^2+1)}dx$.

Sol. Let $I = \int \frac{x^2}{(x-1)(x^2+1)}dx$

Consider $\frac{x^2}{(x-1)(x^2+1)} = \frac{A}{x-1} + \frac{2Bx}{x^2+1} + \frac{C}{x^2+1}$ $\Rightarrow x^2 = A(x^2+1) + 2Bx(x-1) + C(x-1)$

On comparing the coefficients of like terms both sides, we get : $A = 1/2,\ B = 1/4,\ C = 1/2$

So, $I = \int \left(\frac{1}{2} \times \frac{1}{x-1} + \frac{1}{4} \times \frac{2x}{x^2+1} + \frac{1}{2} \times \frac{1}{x^2+1} \right) dx$

$\Rightarrow I = \frac{1}{2}\log|x-1| + \frac{1}{4}\log|x^2+1| + \frac{1}{2}\tan^{-1}x + C$.

Q19. Find the general solution of the following differential equation :

$x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$.

Sol. Here $x\cos\left(\frac{y}{x}\right)\frac{dy}{dx} = y\cos\left(\frac{y}{x}\right) + x$ $\Rightarrow \frac{dy}{dx} = \frac{y}{x} + \sec\left(\frac{y}{x}\right)$ ...(i)

Put $y = vx$ $\Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$.

By (i), we get : $v + x\frac{dv}{dx} = \frac{vx}{x} + \sec\left(\frac{vx}{x}\right)$ $\Rightarrow x\frac{dv}{dx} = \sec v$

$\Rightarrow \int \cos v\,dv = \int \frac{dx}{x}$ $\Rightarrow \sin v = \log|x| + C$ or, $\sin\left(\frac{y}{x}\right) = \log|x| + C$

Therefore, the required solution is $y = x\sin^{-1}\left(\log|x| + C\right)$.

Q20. If four points A, B, C and D with position vectors $4\hat{i}+3\hat{j}+3\hat{k}$, $5\hat{i}+x\hat{j}+7\hat{k}$, $5\hat{i}+3\hat{j}$ and $7\hat{i}+6\hat{j}+\hat{k}$ respectively are coplanar, then find the value of x.

Sol. Let $A(4\hat{i}+3\hat{j}+3\hat{k})$, $B(5\hat{i}+x\hat{j}+7\hat{k})$, $C(5\hat{i}+3\hat{j})$ and $D(7\hat{i}+6\hat{j}+\hat{k})$

$\therefore \vec{a} = \overrightarrow{OB} - \overrightarrow{OA} = \hat{i} + (x-3)\hat{j} + 4\hat{k},\ \vec{b} = \overrightarrow{OC} - \overrightarrow{OB} = (3-x)\hat{j} - 7\hat{k},\ \vec{c} = \overrightarrow{OD} - \overrightarrow{OC} = 2\hat{i} + 3\hat{j} + \hat{k}$

As the points are coplanar so, $[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & x-3 & 4 \\ 0 & 3-x & -7 \\ 2 & 3 & 1 \end{vmatrix} = 0$

$\Rightarrow 1(3-x+21) - (x-3)(0+14) + 4(0-6+2x) = 0$ $\Rightarrow x = 6$ ..

Q21. Find the value of p so that the lines $\frac{1-x}{3} = \frac{7y-14}{2p} = \frac{z-3}{1}$ and $\frac{7-7x}{3p} = \frac{5-y}{1} = \frac{11-z}{7}$ are at right angles.

OR Find the equation of the plane through the line of intersection of the planes $x+y+z=1$ and $2x+3y+4z=5$ and twice of its y-intercept is equal to three times its z-intercept.

Sol. Given lines can be re-written as $\frac{x-1}{-3} = \frac{y-2}{\frac{2p}{7}} = \frac{z-3}{1}$ and $\frac{x-1}{-\frac{3p}{7}} = \frac{y-5}{-1} = \frac{z-11}{-7}$.

The d.r.'s of these lines are $-3, \frac{2p}{7}, 1$ and $-\frac{3p}{7}, -1, -7$ respectively.

As the lines are perpendicular so, by using $a_1a_2 + b_1b_2 + c_1c_2 = 0$ we get :

$(-3)\left(-\frac{3p}{7}\right)+\left(\frac{2p}{7}\right)(-1)+1(-7)=0 \quad \Rightarrow \frac{9p}{7}-\frac{2p}{7}-7=0 \qquad \therefore p=7$.

OR Required plane is $\pi: x+y+z-1+\lambda(2x+3y+4z-5)=0$

That is, $\pi: x(1+2\lambda)+y(1+3\lambda)+z(1+4\lambda)=1+5\lambda \quad \ldots(i)$

So, $\pi: x\left(\frac{1}{\frac{1+5\lambda}{1+2\lambda}}\right)+y\left(\frac{1}{\frac{1+5\lambda}{1+3\lambda}}\right)+z\left(\frac{1}{\frac{1+5\lambda}{1+4\lambda}}\right)=1$

As twice of y-intercept of plane π is equal to three times its z-intercept.

Therefore, $2\left(\frac{1+5\lambda}{1+3\lambda}\right)=3\left(\frac{1+5\lambda}{1+4\lambda}\right) \quad \Rightarrow 2(1+5\lambda)(1+4\lambda)-3(1+5\lambda)(1+3\lambda)=0$

$\Rightarrow (1+5\lambda)\{2+8\lambda-3-9\lambda\}=0 \qquad \therefore \lambda=-1 \quad [\because 1+5\lambda \neq 0$

By (i), required plane is $x+2y+3z=4$.

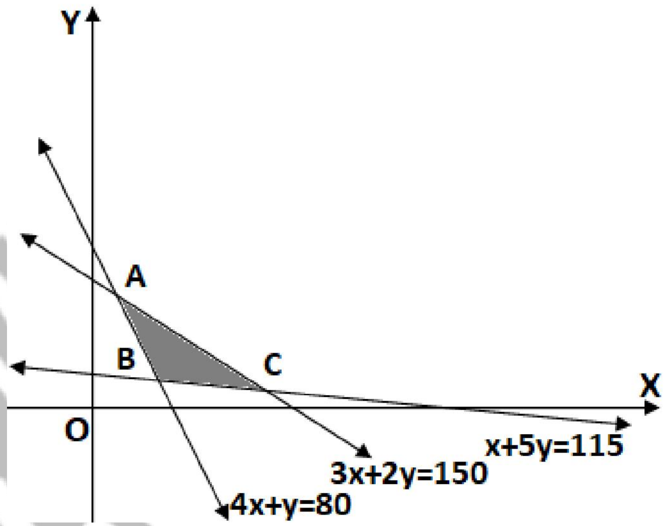
Q22. Solve the following Linear Programming problem graphically :

Minimize : $z=6x+3y$

Subject to the constraints : $\begin{cases} 4x+y \geq 80 \\ x+5y \geq 115 \\ 3x+2y \leq 150 \\ x \geq 0,\ y \geq 0 \end{cases}$.

Sol. Minimize : $z=6x+3y$

Subject to the constraints : $\begin{cases} 4x+y \geq 80 \\ x+5y \geq 115 \\ 3x+2y \leq 150 \\ x \geq 0,\ y \geq 0 \end{cases}$.



Corner Points	Value of z
A (2, 72)	228
B (15, 20)	150 ← Min. value
C (40, 15)	285

So, clearly minimum value of z is 150 at $x=15,\ y=20$.

Q23. There are three categories of students in a class of 60 students :

A : Very hard working students
B : Regular but not so hard working
C : Careless and irregular

10 students are in category A, 30 in category B and rest in category C. It is found that probability of students of category A, unable to get good marks in the final year examination is, 0.002, of category B it is 0.02 and of category C, this probability is 0.20. A student selected at random was found to be the one who could not get good marks in the examination. Find the probability that this student is of category C. What values need to be developed in students of category C?

Sol. Let E : the student is unable to get good marks in the examination.

Also given that n(A) = 10, n(B) = 30, n(C) = 20.

$\therefore P(A)=\frac{10}{60},\ P(B)=\frac{30}{60},\ P(C)=\frac{20}{60},\ P(E|A)=0.002,\ P(E|B)=0.02 \text{ and } P(E|C)=0.20$.

By Bayes' Theorem,

$$P(C|E)=\frac{P(E|C)P(C)}{P(E|A)P(A)+P(E|B)P(B)+P(E|C)P(C)}=\frac{0.20\times\frac{20}{60}}{0.002\times\frac{10}{60}+0.02\times\frac{30}{60}+0.20\times\frac{20}{60}}.$$

i.e., $P(C|E)=\frac{400}{462}=\frac{200}{231}$

Value to be developed in Category C students :
These students should be regular and hard-working for getting success in the examinations.

SECTION – D

Q24. Let $f:\mathbb{R}-\left\{-\frac{4}{3}\right\}\to\mathbb{R}$ be a function defined as $f(x)=\frac{4x}{3x+4}$. Show that, in $f:\mathbb{R}-\left\{-\frac{4}{3}\right\}\to$ Range of f, f is one-one and onto. Hence find f^{-1} in Range of $f\to\mathbb{R}-\left\{-\frac{4}{3}\right\}$.

OR Let $A=\mathbb{R}\times\mathbb{R}$ and * be a binary operation on A defined by (a, b) * (c, d) = (a + c, b + d). Show that * is commutative and associative. Find the identity element for * on A, if any.

Sol. Let $A=\mathbb{R}-\left\{-\frac{4}{3}\right\}$. So, $f:A\to\mathbb{R}$ given by $f(x)=\frac{4x}{3x+4}$.

Let $f(x_1)=f(x_2)$ for $x_1,x_2\in A$ $\Rightarrow\frac{4x_1}{3x_1+4}=\frac{4x_2}{3x_2+4}$ $\Rightarrow 3x_1x_2+4x_1=3x_1x_2+4x_2$

$\Rightarrow x_1=x_2$. So, f (x) is one-one.

Let $y=f(x)=\frac{4x}{3x+4}$, $y\in\mathbb{R}$ $\Rightarrow 3xy+4y=4x$ $\Rightarrow x(4-3y)=4y$ $\Rightarrow x=\frac{4y}{4-3y}\in A$.

As $x\in\mathbb{R}-\left\{-\frac{4}{3}\right\}$ for all $y\in\mathbb{R}$ which implies, Range = codomain so, f is onto.

Hence f^{-1} is $\frac{4y}{4-3y}$ i.e., $f^{-1}(x)=\frac{4x}{4-3x}$.

OR Here $A=\mathbb{R}\times\mathbb{R}$ and * is a binary operation on A defined by
$(a,b)*(c,d)=(a+c,\ b+d)$.
Commutativity : Let $(a,b),(c,d)\in A$.
Then $(a,b)*(c,d)=(a+c,\ b+d)=(c+a,\ d+b)=(c,d)*(a,b)$.
Hence, $(a,b)*(c,d)=(c,d)*(a,b)$. $\therefore$ * is commutative.
Associativity : Let $(a,b),(c,d),(e,f)\in A$. Then we have,

$[(a,b)*(c,d)]*(e,f)=(a+c,\ b+d)*(e,f)=(a+c+e,\ b+d+f)=\left((a+c)+e,(b+d)+f\right)$

$\Rightarrow \quad =\left(a+(c+e),b+(d+f)\right)$

$\Rightarrow \quad =(a,b)*(c+e,d+f)=(a,b)*[(c,d)*(e,f)]$

Hence, $[(a,b)*(c,d)]*(e,f)=(a,b)*[(c,d)*(e,f)]$. $\therefore$ * is associative.
Let (x, y) be identity element for * on A.
Then $(a,b)*(x,y)=(a,b)$ $\Rightarrow(a+x,b+y)=(a,b)$
$\Rightarrow a+x=a,\ b+y=b$ $\Rightarrow x=0,\ y=0$.
So, the identity element for the binary operation * is (0, 0).

Q25. Find matrix A, if $\begin{pmatrix}2&-1\\1&0\\-3&4\end{pmatrix}A=\begin{pmatrix}-1&-8&-10\\1&-2&-5\\9&22&15\end{pmatrix}$.

Sol. Let $A=\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$. $\{\because [P]_{3\times2}[A]_{m\times n}=[Q]_{3\times3}\ \therefore m=2,\ n=3$

As $\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}A=\begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$

So, $\begin{pmatrix} 2 & -1 \\ 1 & 0 \\ -3 & 4 \end{pmatrix}\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}=\begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 2a-d & 2b-e & 2c-f \\ a & b & c \\ -3a+4d & -3b+4e & -3c+4f \end{pmatrix}=\begin{pmatrix} -1 & -8 & -10 \\ 1 & -2 & -5 \\ 9 & 22 & 15 \end{pmatrix}$

By equality of matrices, we get :

$2a-d=-1,\ 2b-e=-8,\ 2c-f=-10,\ a=1,\ b=-2,\ c=-5,$

$-3a+4d=9,\ -3b+4e=22,\ -3c+4f=15$

So, $a=1,\ b=-2,\ c=-5,\ d=3,\ e=4,\ f=0$.

Therefore, $A=\begin{pmatrix} 1 & -2 & -5 \\ 3 & 4 & 0 \end{pmatrix}$.

Q26. Find the intervals in which the function f given by $f(x)=\sin x+\cos x,\ 0\le x\le 2\pi$ is strictly increasing or strictly decreasing.

OR Show that the height of the cylinder of greatest volume which can be inscribed in a right circular cone of height h and semi-vertical angle α, is one-third that of the cone. Hence find the greatest volume of the cylinder.

Sol. Given $f(x)=\sin x+\cos x,\ 0\le x\le 2\pi$ $\Rightarrow f'(x)=\cos x-\sin x$

For critical points, $f'(x)=\sin x-\cos x=0$ $\Rightarrow \tan x=1\quad \therefore x=\pi/4, 5\pi/4$

Interval	Sign of $f'(x)$	$f(x)$ is strictly
$(0,\pi/4)$	Positive	Increasing
$(\pi/4,5\pi/4)$	Negative	Decreasing
$(5\pi/4,2\pi)$	Positive	Increasing

OR Let height and radius of the cylinder inscribed in the cone be H and R respectively.

Let $\angle AOC=\alpha$. Since $\Delta AOC \sim \Delta AO'B$

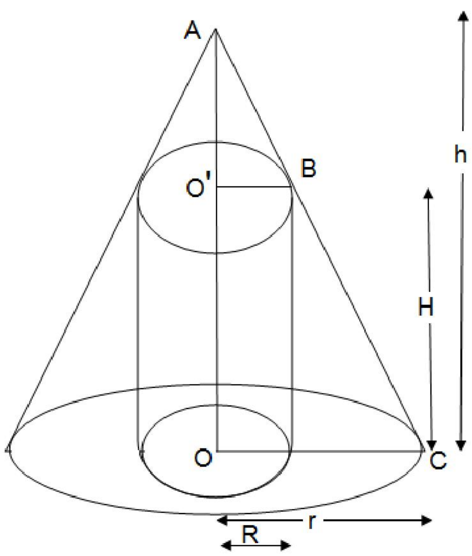
$\therefore \dfrac{AO}{AO'}=\dfrac{OC}{O'B}\ \Rightarrow \dfrac{h}{h-H}=\dfrac{r}{R}$ i.e., $H=\left(1-\dfrac{R}{r}\right)h$

Also in ΔAOC, $\tan\alpha=\dfrac{OC}{OA}$ $\Rightarrow r=h\tan\alpha$

$\therefore H=h\left(1-\dfrac{R}{h\tan\alpha}\right)\ldots$(i)

Now volume of cylinder, $V=\pi R^2H=\pi R^2h\left(1-\dfrac{R}{h\tan\alpha}\right)$

$\Rightarrow V=\pi h\left(R^2-\dfrac{R^3}{h\tan\alpha}\right)$



$$\Rightarrow \frac{dV}{dR} = \pi h\left(2R - \frac{3R^2}{h\tan\alpha}\right) \text{ and } \frac{d^2V}{dR^2} = \pi h\left(2 - \frac{6R}{h\tan\alpha}\right)$$

For local points of maxima and/or minima, $\frac{dV}{dR} = \pi h\left(2R - \frac{3R^2}{h\tan\alpha}\right) = 0$

$$\Rightarrow 2R - \frac{3R^2}{h\tan\alpha} = 0 \qquad \Rightarrow R = \frac{2h\tan\alpha}{3}$$

$$\because \left.\frac{d^2V}{dR^2}\right]_{\text{at } R=\frac{2h\tan\alpha}{3}} = \pi h\left(2 - \frac{6}{h\tan\alpha} \times \frac{2h\tan\alpha}{3}\right) = -2\pi h < 0$$

$\therefore$ V is maximum at $R = \frac{2h\tan\alpha}{3}$

Now height of the cylinder, $H = h\left(1 - \frac{R}{h\tan\alpha}\right) = h\left(1 - \frac{2}{3}\right)$ $\qquad \therefore H = \frac{h}{3}$

Also volume of the cylinder, $V = \pi\left(\frac{2h\tan\alpha}{3}\right)^2 h\left(1 - \frac{2}{3}\right)$

$$\Rightarrow V = \frac{\pi h}{3}\left(\frac{4h^2\tan^2\alpha}{9}\right) \qquad \therefore V = \frac{4\pi h^3\tan^2\alpha}{27}.$$

Q27. Using integration find the area of region $\left\{(x,y) : y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\right\}$.

Sol. We have $\left\{(x,y) : y^2 \le 4x,\ 4x^2 + 4y^2 \le 9\right\}$

Consider $y^2 = 4x$...(i), $4x^2 + 4y^2 = 9$...(ii)

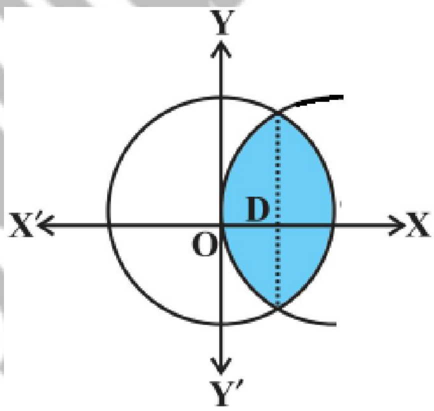
Solving (i) & (ii), we get $4x^2 + 16x = 9$

$\Rightarrow 4x^2 + 16x - 9 = 0 \qquad \Rightarrow (2x-1)(2x+9) = 0$

$\Rightarrow x = \frac{1}{2}, -\frac{9}{2}$

$\therefore$ Point of intersection : $P(1/2, \pm\sqrt{2})$.

Note that (i) is symmetrical about x-axis and (ii) is symmetrical about both the axes.



$$\therefore \text{requierd area} = 2\left(\int_0^{1/2} y_i\,dx + \int_{1/2}^{3/2} y_{ii}\,dx\right)$$

$$\Rightarrow \quad = 2\left(2\int_0^{1/2}\sqrt{x}\,dx + \int_{1/2}^{3/2}\sqrt{\left(\frac{3}{2}\right)^2 - x^2}\,dx\right)$$

$$\Rightarrow \quad = 2\left(\frac{4}{3}\left[x^{3/2}\right]_0^{1/2} + \left[\frac{x}{2}\sqrt{\frac{9}{4} - x^2} + \frac{9}{8}\sin^{-1}\frac{x}{3/2}\right]_{1/2}^{3/2}\right)$$

$$\Rightarrow \quad = \frac{8}{3}\left[\frac{1}{2\sqrt{2}}\right] + 2\left\{\left[0 + \frac{9}{8}\times\frac{\pi}{2}\right] - \left[\frac{\sqrt{2}}{4} + \frac{9}{8}\sin^{-1}\frac{1}{3}\right]\right\}$$

$$\Rightarrow \quad = \frac{4}{3\sqrt{2}} + \left\{\frac{9\pi}{8} - \frac{\sqrt{2}}{2} - \frac{9}{4}\sin^{-1}\frac{1}{3}\right\} = \frac{4}{3\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{9}{4}\left\{\frac{\pi}{2} - \sin^{-1}\frac{1}{3}\right\}$$

$$\Rightarrow \quad = \left(\frac{1}{3\sqrt{2}} + \frac{9}{4}\cos^{-1}\frac{1}{3}\right) \text{Sq.units}.$$

Q28. Find the equation of plane containing lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ and $\frac{x-38}{3}=\frac{y+29}{8}=\frac{z-5}{-5}$.

Sol. The lines $\frac{x-8}{3}=\frac{y+19}{-16}=\frac{z-10}{7}$ passes through (8, –19, 10).

Also the d.r.'s of the given lines are 3, –16, 7 and 3, 8, –5.

So, the d.r.'s of the plane containing these lines (i.e., the d.r.'s of the normal to the plane containing lines) can be obtained by $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix} = 24\hat{i}+36\hat{j}+72\hat{k}$.

Therefore the required plane is

$24(x-8)+36(y+19)+72(z-10)=0$ i.e., $2(x-8)+3(y+19)+6(z-10)=0$

$\therefore 2x+3y+6z=19$.

Q29. A fair coin is tossed 8 times, find the probability of

(i) exactly 5 heads

(ii) at least six heads

(iii) at most six heads.

OR Three cards are drawn successively with replacement from a well shuffled pack of 52 cards. Find the mean and variance of the number of red cards.

Sol. Let E : getting a head so, P(E) = p = 1/2, q = 1 – p = 1/2.

Here n = 8.

So, $P(X=r)={}^{8}C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{8-r}=\frac{1}{2^8}\times {}^{8}C_r$.

(i) $P(X=5)=\frac{1}{2^8}\times {}^{8}C_5=\frac{56}{256}$

(ii) $P(X\ge 6)=P(X=6)+P(X=7)+P(X=8)=\frac{1}{2^8}\times\left({}^{8}C_6+{}^{8}C_7+{}^{8}C_8\right)=\frac{1}{256}(28+8+1)=\frac{37}{256}$.

(iii) $P(X\le 6)=1-P(X>6)=1-[P(X=7)+P(X=8)]=1-\frac{1}{2^8}\times\left({}^{8}C_7+{}^{8}C_8\right)=1-\frac{9}{256}=\frac{247}{256}$.

OR Let X : number of red cards. So X can take values 0, 1, 2, 3.

Let E : getting red cards. So, $P(E)=\frac{26}{52}=\frac{1}{2}$, $P(\bar{E})=1-\frac{1}{2}=\frac{1}{2}$.

X	0	1	2	3
P(X)	$P(\bar{E})P(\bar{E})P(\bar{E})=\frac{1}{8}$	$3P(E)P(\bar{E})P(\bar{E})=\frac{3}{8}$	$3P(E)P(E)P(\bar{E})=\frac{3}{8}$	$P(E)P(E)P(E)=\frac{1}{8}$

Now, Mean, $\mu=\sum X\,P(X)=\frac{12}{8}=\frac{3}{2}$,

And Variance, $\sigma^2=\sum X^2\,P(X)-\mu^2=\frac{24}{8}-\left(\frac{3}{2}\right)^2=3-\frac{9}{4}=\frac{3}{4}$.