CHALLENGE 30 ON DEFINITE INTEGRALS

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Q01. Let $\mathrm{I}=\int_0^{\pi/2}\frac{1}{2-\sin x}dx=\int_0^{\pi/2}\frac{1}{2-\frac{2\tan(x/2)}{1+\tan^2(x/2)}}dx=\int_0^{\pi/2}\frac{\sec^2(x/2)}{2+2\tan^2(x/2)-2\tan(x/2)}dx$

Put $\tan(x/2)=t\Rightarrow\sec^2(x/2)dx=2dt$. Also *when* $x=0\Rightarrow t=0$ & *when* $x=\pi/2\Rightarrow t=1$

$\therefore\mathrm{I}=\int_0^1\frac{dt}{t^2-t+1}=\int_0^1\frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$ $\quad\Rightarrow\mathrm{I}=\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{2t-1}{\sqrt{3}}\right)\right]_0^1$

$\Rightarrow\mathrm{I}=\frac{2}{\sqrt{3}}\left[\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)-\tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right]$ $\quad\therefore\mathrm{I}=\frac{2\pi}{3\sqrt{3}}$

Q02. Let $\mathrm{I}=\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{3/2}}dx=\int_{\pi/3}^{\pi/2}\frac{\sqrt{1+\cos x}}{(1-\cos x)^{3/2}}\times\frac{\sqrt{1-\cos x}}{\sqrt{1-\cos x}}dx=\int_{\pi/3}^{\pi/2}\frac{\sin x}{(1-\cos x)^2}dx$

Put $1-\cos x=t\Rightarrow\sin x\,dx=dt$. Also *when* $x=\pi/3\Rightarrow t=1/2$ & *when* $x=\pi/2\Rightarrow t=1$

$\therefore\mathrm{I}=\int_{1/2}^1\frac{dt}{t^2}$ $\quad\Rightarrow\mathrm{I}=-\left[\frac{1}{t}\right]_{1/2}^1=-\left[\frac{1}{1}-\frac{1}{1/2}\right]=1$.

Q03. Let $\mathrm{I}=\int_0^1\frac{x}{(1-x)^{3/4}}dx$

Put $1-x=t^{4/3}\Rightarrow dx=-\frac{4}{3}t^{1/3}dt$. Also *when* $x=0\Rightarrow t=1$ & *when* $x=1\Rightarrow t=0$

$\therefore\ \mathrm{I}=-\frac{4}{3}\int_1^0\left(\frac{1-t^{4/3}}{t}\right)t^{1/3}dt=\frac{4}{3}\int_0^1\left(\frac{1}{t^{2/3}}-t^{2/3}\right)dt=\frac{4}{3}\left[3t^{1/3}-\frac{3}{5}t^{5/3}\right]_0^1$

$\Rightarrow\mathrm{I}=4\left[\left(1-\frac{1}{5}\right)-0\right]$ $\quad\therefore\mathrm{I}=\frac{16}{5}$.

Q04. Let $\mathrm{I}=\int_0^{\pi/2}\frac{\sec^2x}{\sec^2x+\mathrm{cosec}^2x}dx$ ...(i)

$\Rightarrow\mathrm{I}=\int_0^{\pi/2}\frac{\sec^2\left(\frac{\pi}{2}-x\right)}{\sec^2\left(\frac{\pi}{2}-x\right)+\mathrm{cosec}^2\left(\frac{\pi}{2}-x\right)}dx$ $\quad\Rightarrow\mathrm{I}=\int_0^{\pi/2}\frac{\mathrm{cosec}^2x}{\mathrm{cosec}^2x+\sec^2x}dx$ ...(ii)

Adding (i) and (ii), $2\mathrm{I}=\int_0^{\pi/2}\left(\frac{\sec^2x}{\sec^2x+\mathrm{cosec}^2x}+\frac{\mathrm{cosec}^2x}{\mathrm{cosec}^2x+\sec^2x}\right)dx$ $\quad\Rightarrow 2\mathrm{I}=\int_0^{\pi/2}1\,dx$

$\Rightarrow 2\mathrm{I}=\left[x\right]_0^{\pi/2}=\frac{\pi}{2}-0$ $\quad\therefore\mathrm{I}=\frac{\pi}{4}$.

Q05. Let $\mathrm{I}=\int_0^{\pi}\frac{x}{1+\cos\alpha\sin x}dx,\ 0<\alpha<\pi$ ...(i)

$\Rightarrow\mathrm{I}=\int_0^{\pi}\frac{\pi-x}{1+\cos\alpha\sin(\pi-x)}dx$ $\quad\Rightarrow\mathrm{I}=\int_0^{\pi}\frac{\pi-x}{1+\cos\alpha\sin x}dx$ ...(ii)

Adding (i) & (ii), $2\mathrm{I}=\int_0^{\pi}\left(\frac{x}{1+\cos\alpha\sin x}+\frac{\pi-x}{1+\cos\alpha\sin x}\right)dx$ $\quad\Rightarrow\mathrm{I}=\frac{\pi}{2}\int_0^{\pi}\frac{1}{1+\cos\alpha\sin x}dx$

$$\Rightarrow \mathrm{I} = \frac{\pi}{2}\int_0^{\pi} \frac{\sec^2(x/2)}{1+\tan^2(x/2)+2\cos\alpha\tan(x/2)}dx$$

Put $\tan(x/2) = t \Rightarrow \sec^2(x/2)dx = 2dt$. Also *when* $x = 0 \Rightarrow t = 0$ & *when* $x = \pi/2 \Rightarrow t = \infty$

$$\therefore \mathrm{I} = \frac{\pi}{2}\int_0^{\infty} \frac{2dt}{t^2+2t\cos\alpha+1} = \pi\int_0^{\infty}\frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha} = \frac{\pi}{\sin\alpha}\left[\tan^{-1}\frac{t+\cos\alpha}{\sin\alpha}\right]_0^{\infty}$$

$$\Rightarrow \mathrm{I} = \frac{\pi}{\sin\alpha}\left[\frac{\pi}{2}-\left(\frac{\pi}{2}-\alpha\right)\right] \qquad \therefore \mathrm{I} = \frac{\pi\,\alpha}{\sin\alpha}.$$

Q06. Let $\mathrm{I} = \int_{-1/2}^{1/2}\left|x\cos\frac{\pi x}{2}\right|dx$

$\left\{\begin{array}{l}\text{Consider } f(x) = \left|x\cos\frac{\pi x}{2}\right| \Rightarrow f(-x) = \left|-x\cos\frac{-\pi x}{2}\right| = \left|x\cos\frac{\pi x}{2}\right| \\ \because f(x) = f(-x) \ \therefore f(x) \text{ is an even function}\end{array}\right.$

$$\text{So, } \mathrm{I} = 2\int_0^{1/2}\left|x\cos\frac{\pi x}{2}\right|dx = 2\int_0^{1/2}x\cos\frac{\pi x}{2}dx \text{ ...(i)}$$

Consider $2\int x\cos\frac{\pi x}{2}dx = 2\left[x\int\cos\frac{\pi x}{2}dx - \int\left(\frac{d}{dx}x\int\cos\frac{\pi x}{2}dx\right)dx\right] = 2\left[\frac{2x}{\pi}\sin\frac{\pi x}{2}+\frac{4}{\pi^2}\cos\frac{\pi x}{2}\right]$

Substituting value of integral in (i), $\mathrm{I} = 2\left[\frac{2x}{\pi}\sin\frac{\pi x}{2}+\frac{4}{\pi^2}\cos\frac{\pi x}{2}\right]_0^{1/2}$

$$\Rightarrow \mathrm{I} = 2\left[\left(\frac{1}{\pi}\sin\frac{\pi}{4}+\frac{4}{\pi^2}\cos\frac{\pi}{4}\right)-\left(0+\frac{4}{\pi^2}\cos 0\right)\right]$$

$$\Rightarrow \mathrm{I} = 2\left[\frac{1}{\sqrt{2}\,\pi}+\frac{4}{\sqrt{2}\,\pi^2}-\frac{4}{\pi^2}\right] \qquad \therefore \mathrm{I} = \frac{\sqrt{2}}{\pi^2}\left(\pi+4-4\sqrt{2}\right).$$

Q07. Let $\mathrm{I} = \int_0^{2\pi}\left(\sin x+|\sin x|\right)dx = \int_0^{\pi}\left(\sin x+|\sin x|\right)dx + \int_{\pi}^{2\pi}\left(\sin x+|\sin x|\right)dx$

$$\Rightarrow \mathrm{I} = \int_0^{\pi}2\sin x dx + \int_{\pi}^{2\pi}\left(\sin x-\sin x\right)dx = -2\left[\cos x\right]_0^{\pi} = -2\left[-1-1\right] \qquad \therefore \mathrm{I} = 4.$$

Q08. Let $\mathrm{I} = \int_0^1\frac{dx}{x^2+2x\cos\alpha+1} = \int_0^1\frac{dx}{(x+\cos\alpha)^2+\sin^2\alpha} = \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)\right]_0^1$

$$\Rightarrow \quad = \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right)-\tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right] = \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\cot\frac{\alpha}{2}\right)-\tan^{-1}\left(\cot\alpha\right)\right]$$

$$\Rightarrow \quad = \frac{1}{\sin\alpha}\left[\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right] \qquad \therefore \mathrm{I} = \frac{\alpha}{2\sin\alpha}.$$

Q09. Let $\mathrm{I} = \int_0^m\frac{x^4}{\sqrt{m^2-x^2}}dx$

Put $x = m\sin\theta \Rightarrow dx = m\cos\theta d\theta$. Also *when* $x = 0 \Rightarrow \theta = 0$ & *when* $x = m \Rightarrow t = \pi/2$

$$\Rightarrow \mathrm{I} = \int_0^{\pi/2}\frac{m^4\sin^4\theta}{\sqrt{m^2-m^2\sin^2\theta}}m\cos\theta d\theta = \int_0^{\pi/2}m^4\sin^4\theta\, d\theta \qquad \Rightarrow \mathrm{I} = m^4\int_0^{\pi/2}\left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta$$

$$\Rightarrow \mathrm{I} = \frac{m^4}{4}\int_0^{\pi/2}\left(1-2\cos 2\theta+\cos^2 2\theta\right)d\theta = \frac{m^4}{4}\int_0^{\pi/2}\left(1-2\cos 2\theta+\frac{1+\cos 4\theta}{2}\right)d\theta$$

$$\Rightarrow \mathrm{I} = \frac{m^4}{4}\int_0^{\pi/2}\left(\frac{3}{2}-2\cos 2\theta+\frac{1}{2}\cos 4\theta\right)d\theta = \frac{m^4}{4}\left[\frac{3}{2}\theta-\sin 2\theta+\frac{1}{8}\sin 4\theta\right]_0^{\pi/2}$$

$$\Rightarrow I=\frac{m^4}{4}\left[\left(\frac{3\pi}{4}-\sin\pi+\frac{1}{8}\sin 2\pi\right)-0\right]=3\pi\frac{m^4}{16} \qquad \therefore I=3\pi\left(\frac{m}{2}\right)^4.$$

Q10. Let $I=\int\frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2}$ [On dividing Nr & Dr by $\cos^4x$

$$\Rightarrow I=\int\frac{(\sec^2x.\sec^2x)\,dx}{(a^2+b^2\tan^2x)^2}$$

Put $b\tan x = a\tan\theta \Rightarrow \sec^2x\,dx=\frac{a}{b}\sec^2\theta\,d\theta$

$$\Rightarrow I=\int\frac{(1+\tan^2x)\frac{a}{b}(1+\tan^2\theta)\,d\theta}{(a^2+a^2\tan^2\theta)^2} \qquad \Rightarrow I=\frac{1}{a^3b}\int\frac{(1+\tan^2x)\,d\theta}{(1+\tan^2\theta)}$$

$$\Rightarrow I=\frac{1}{a^3b}\int\left(1+\frac{a^2}{b^2}\tan^2\theta\right)\frac{d\theta}{\sec^2\theta} \qquad \Rightarrow I=\frac{1}{a^3b^3}\int\left(b^2\cos^2\theta+a^2\sin^2\theta\right)d\theta$$

$$\Rightarrow I=\frac{1}{a^3b^3}\int\left(b^2\frac{1+\cos 2\theta}{2}+a^2\frac{1-\cos 2\theta}{2}\right)d\theta \Rightarrow I=\frac{1}{2a^3b^3}\int[(b^2+a^2)+(b^2-a^2)\cos 2\theta]\,d\theta$$

$$\Rightarrow I=\frac{1}{2a^3b^3}\left[(b^2+a^2)\theta+\frac{(b^2-a^2)}{2}\sin 2\theta\right] \qquad \left[\text{When } x=0,\theta=0 \text{ and when } x=\frac{\pi}{2},\theta=\frac{\pi}{2}\right.$$

$$\therefore \int_0^{\frac{\pi}{2}}\frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2}=\frac{1}{2a^3b^3}\left[(b^2+a^2)\theta+\frac{(b^2-a^2)}{2}\sin 2\theta\right]_0^{\frac{\pi}{2}}$$

So, $\int_0^{\frac{\pi}{2}}\frac{dx}{(a^2\cos^2x+b^2\sin^2x)^2}=\frac{1}{2a^3b^3}\left[(b^2+a^2)\frac{\pi}{2}\right]=\frac{\pi(b^2+a^2)}{4a^3b^3}$.

Q11. Let $I=\int_0^{\pi/2}\log\sec x\,dx=-\int_0^{\pi/2}\log\cos x\,dx$

Now refer Example 36 Chapter 07 NCERT Part II, $\because \int_0^{\pi/2}\log\sin x\,dx=\int_0^{\pi/2}\log\cos x\,dx=-\frac{\pi}{2}\log 2$

$\therefore\ I=-\left(-\frac{\pi}{2}\log 2\right)=\frac{\pi}{2}\log 2$.

Q12. Let $I=\int_{-2}^{2}\frac{dx}{1+|x-1|}=\int_{-2}^{1}\frac{dx}{1+|x-1|}+\int_{1}^{2}\frac{dx}{1+|x-1|} \qquad \Rightarrow I=\int_{-2}^{1}\frac{dx}{1-(x-1)}+\int_{1}^{2}\frac{dx}{1+(x-1)}$

$$\Rightarrow I=-\left[\log|2-x|\right]_{-2}^{1}+\left[\log|x|\right]_1^2=-\left[0-\log 4\right]+\left[\log 2-0\right] \qquad \therefore I=3\log 2.$$

Q13. Let $I=\int_0^{\pi}\log(1+\cos x)\,dx$ ...(i)

$$\Rightarrow I=\int_0^{\pi}\log\left(1+\cos(\pi-x)\right)dx \qquad \Rightarrow I=\int_0^{\pi}\log(1-\cos x)\,dx \text{ ...(ii)}$$

Adding (i) and (ii), we get : $2I=\int_0^{\pi}\left[\log(1-\cos x)+\log(1+\cos x)\right]dx \qquad \Rightarrow 2I=2\int_0^{\pi}\log\sin x\,dx$

$$\Rightarrow I=\int_0^{\pi}\log\sin x\,dx \qquad \Rightarrow I=2\int_0^{\pi/2}\log\sin x\,dx$$

[Now refer Example 36 Chapter 07 NCERT Part II, $\because \int_0^{\pi/2}\log\sin x\,dx=-\frac{\pi}{2}\log 2$]

$$\therefore I=2\left[-\frac{\pi}{2}\log 2\right]=\pi\log\left(\frac{1}{2}\right)$$

Q14. Let $\mathrm{I}=\int_0^{\pi/2}\frac{\cos x}{1+\cos x+\sin x}dx=\int_0^{\pi/2}\frac{\cos^2(x/2)-\sin^2(x/2)}{2\cos^2(x/2)+2\sin(x/2)\cos(x/2)}dx$

$\Rightarrow \mathrm{I}=\int_0^{\pi/2}\frac{\cos^2(x/2)-\sin^2(x/2)}{2\cos(x/2)\left[\cos(x/2)+\sin(x/2)\right]}dx=\int_0^{\pi/2}\frac{\cos(x/2)-\sin(x/2)}{2\cos(x/2)}dx$

$=\frac{1}{2}\int_0^{\pi/2}\left[1-\tan(x/2)\right]dx=\frac{1}{2}\left[x+2\log\left|\cos(x/2)\right|\right]_0^{\pi/2}=\frac{1}{2}\left[\left(\frac{\pi}{2}+2\log\left|\cos\frac{\pi}{2}\right|\right)-\left(0+2\log 1\right)\right]$

$\therefore \mathrm{I}=\left(\frac{\pi}{4}+\frac{1}{2}\log\frac{1}{2}\right)$.

Q15. Consider $\frac{(2-x^2)}{(1-x)\sqrt{1-x^2}}=\frac{(1-x^2)}{(1-x)\sqrt{1-x^2}}+\frac{1}{(1-x)\sqrt{1-x^2}}=\frac{(1+x)}{\sqrt{1-x^2}}+\frac{1}{(1-x)\sqrt{1-x^2}}$

$\Rightarrow \quad =\sqrt{\frac{1+x}{1-x}}+\frac{1}{(1-x)\sqrt{1-x^2}}$

Using $\int e^x\left[f(x)+f'(x)\right]dx=e^x f(x)+C$, we have :

$\int_{-1}^{1/2}e^x\left[\sqrt{\frac{1+x}{1-x}}+\frac{1}{(1-x)\sqrt{1-x^2}}\right]dx=\left[e^x\sqrt{\frac{1+x}{1-x}}\right]_{-1}^{1/2} \quad \Rightarrow \quad =\left[e^{1/2}\sqrt{3}-0\right]=\sqrt{3e}$.

Q16. Consider $\mathrm{I}=\int\left[f(x)+xf'(x)\right]dx=\int f(x)dx+\int x\,f'(x)dx$

Applying By Parts in second integral, we get :

$\mathrm{I}=\int f(x)dx+x\int f'(x)dx-\int\left(\frac{d}{dx}x\int f'(x)dx\right)dx=\int f(x)dx+xf(x)-\int f(x)dx=xf(x)$

$\therefore \int_0^m\left[f(x)+xf'(x)\right]dx=\left[xf(x)\right]_0^m=mf(m)$.

Q17. Consider $\mathrm{I}=\int_0^m f(-x)dx$

Put $x=-t\Rightarrow dx=-dt$. Also when $x=0\Rightarrow t=0$ & when $x=m\Rightarrow t=-m$

$\therefore \mathrm{I}=\int_0^{-m}f(t)(-dt)=\int_{-m}^0 f(x)dx$

So, $\int_0^m\left[f(x)+f(-x)\right]dx=\int_0^m f(x)dx+\int_0^m f(-x)dx=\int_0^m f(x)dx+\int_{-m}^0 f(x)dx=\int_{-m}^m f(x)dx$

$$\therefore \int_{-m}^m f(x)\,dx=\begin{cases}2\int_0^m f(x)\,dx, \text{ if } f(x) \text{ is even function } i.e.,\ f(-x)=f(x)\\ 0, \text{ if } f(x) \text{ is odd function } i.e.,\ f(-x)=-f(x)\end{cases}.$$

Q18. Let $\mathrm{I}=\int_0^{\pi}x\log\sin x\,dx$ ...(i)

$\Rightarrow \mathrm{I}=\int_0^{\pi}(\pi-x)\log\sin(\pi-x)\,dx \qquad \Rightarrow \mathrm{I}=\int_0^{\pi}(\pi-x)\log\sin x\,dx$ ...(ii)

Adding (i) & (ii), we get : $2\mathrm{I}=\pi\int_0^{\pi}\log\sin x\,dx \qquad \Rightarrow 2\mathrm{I}=2\pi\int_0^{\pi/2}\log\sin x\,dx$

$$\left[\begin{array}{c}\because f(x)=\log\sin x \Rightarrow f(\pi-x)=\log\sin(\pi-x)=\log\sin x \Rightarrow f(\pi-x)=f(x) \\ \text{Using } \int_0^{2a} f(x)\,dx=\begin{cases} 2\int_0^a f(x)\,dx, \text{ if } f(2a-x)=f(x) \\ 0, \text{ if } f(2a-x)=-f(x)\end{cases}\end{array}\right]$$

$$\therefore \mathrm{I}=\pi\int_0^{\pi/2}\log\sin x\,dx=\pi\left[-\frac{\pi}{2}\log 2\right]=\frac{\pi^2}{2}\log\left(\frac{1}{2}\right)$$

Q19. Let $\mathrm{I}=\int_0^{2\pi}\log(1+\sin x)\,dx$ ...(i) $\quad\Rightarrow \mathrm{I}=\int_0^{2\pi}\log(1+\sin[2\pi-x])\,dx=\int_0^{2\pi}\log(1-\sin x)\,dx$ ...(ii)

Adding (i) & (ii), $2\mathrm{I}=\int_0^{2\pi}\left[\log(1+\sin x)+\log(1-\sin x)\right]dx \quad \Rightarrow 2\mathrm{I}=\int_0^{2\pi}\log\cos^2 x\,dx$

$\Rightarrow 2\mathrm{I}=2\int_0^{\pi}\log\cos^2 x\,dx \quad \Rightarrow \mathrm{I}=2\int_0^{\pi/2}\log\cos^2 x\,dx \quad \Rightarrow \mathrm{I}=4\int_0^{\pi/2}\log\cos x\,dx$

$\Rightarrow \mathrm{I}=4\left[-\frac{\pi}{2}\log 2\right] \quad \therefore \mathrm{I}=-2\pi\log 2$.

Q20. Let $\mathrm{I}=\int_0^{\infty}\log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2}$

Put $x=\tan\theta \Rightarrow dx=\sec^2\theta d\theta$. Also when $x=0\Rightarrow\theta=0$ & when $x=\infty\Rightarrow\theta=\frac{\pi}{2}$

$$\therefore \mathrm{I}=\int_0^{\pi/2}\log\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{\sec^2\theta d\theta}{1+\tan^2\theta}=\int_0^{\pi/2}\log\left(\frac{1+\tan^2\theta}{\tan\theta}\right)d\theta=\int_0^{\pi/2}\log\left(\frac{1}{\cos^2\theta}\times\frac{\cos\theta}{\sin\theta}\right)d\theta$$

$$\Rightarrow \mathrm{I}=\int_0^{\pi/2}\log\left(\frac{2}{\sin 2\theta}\right)d\theta=\int_0^{\pi/2}\log 2d\theta-\int_0^{\pi/2}\log\sin 2\theta d\theta \text{ ...(i)}$$

Consider $\mathrm{I}_1=-\int_0^{\pi/2}\log\sin 2\theta d\theta$

Put $2\theta=t\Rightarrow d\theta=\frac{1}{2}dt$. Also when $\theta=0\Rightarrow t=0$ & when $\theta=\frac{\pi}{2}\Rightarrow t=\pi$

$$\therefore \mathrm{I}_1=-\frac{1}{2}\int_0^{\pi}\log\sin t\,dt=-\frac{1}{2}\times 2\int_0^{\pi/2}\log\sin t\,dt=-\left[-\frac{\pi}{2}\log 2\right]=\frac{\pi}{2}\log 2$$

Substituting value of I_1 in (i), we get : $\mathrm{I}=\log 2\left[x\right]_0^{\pi/2}+\frac{\pi}{2}\log 2$

$\mathrm{I}=\log 2\left[\frac{\pi}{2}-0\right]+\frac{\pi}{2}\log 2 \quad \therefore \mathrm{I}=\pi\log 2$.

Q21. Let $\mathrm{I}=\int_{-1}^{1}\frac{\sin x-x^2}{3-|x|}dx=\int_{-1}^{1}\frac{\sin x}{3-|x|}dx-\int_{-1}^{1}\frac{x^2}{3-|x|}dx$

[Note that function in 1st integral is odd function and, in 2nd integral function is even.]

$$\therefore \mathrm{I}=0-2\int_0^1\frac{x^2}{3-|x|}dx=2\int_0^1\frac{x^2}{x-3}dx=2\int_0^1\left(\frac{x^2-9}{x-3}+\frac{9}{x-3}\right)dx=2\int_0^1\left(x+3+\frac{9}{x-3}\right)dx$$

$$\Rightarrow \mathrm{I}=2\left[\frac{1}{2}x^2+3x+9\log|x-3|\right]_0^1=2\left[\left(\frac{1}{2}+3+9\log 2\right)-\left(9\log 3\right)\right]=7+18\log\left(\frac{2}{3}\right)$$

Q22. $\int_3^x\sqrt{x+1}\,dx=0 \quad \Rightarrow \frac{2}{3}\left[(x+1)^{3/2}\right]_3^x=0 \quad \Rightarrow (x+1)^{3/2}-4^{3/2}=0$

$\Rightarrow x+1=4 \qquad \therefore x=3$

Q23. $\int_{\sqrt{2}}^{x}\frac{1}{x\sqrt{x^2-1}}dx=\frac{\pi}{2} \quad \Rightarrow\left[\sec^{-1}x\right]_{\sqrt{2}}^{x}=\frac{\pi}{2} \quad \Rightarrow \sec^{-1}x-\sec^{-1}\sqrt{2}=\frac{\pi}{2}$

$\Rightarrow x=\sec\left(\frac{\pi}{2}+\frac{\pi}{4}\right)=-\text{cosec}\left(\frac{\pi}{4}\right) \qquad \therefore x=-\sqrt{2}$.

Q24. Put $x+\sqrt{1+x^2}=t \Rightarrow 1+x^2=(t-x)^2 \Rightarrow 1=t^2-2tx \Rightarrow x=\frac{1}{2}\left(t-\frac{1}{t}\right) \Rightarrow dx=\frac{1}{2}\left(1+\frac{1}{t^2}\right)dt$

Also when $x=0 \Rightarrow t=1$ and when $x=\infty \Rightarrow t=\infty$

Let $I=\int_0^{\infty}\frac{dx}{[x+\sqrt{1+x^2}]^n} \qquad \Rightarrow I=\frac{1}{2}\int_1^{\infty}\frac{dt}{t^n}\left(1+\frac{1}{t^2}\right)=\frac{1}{2}\int_1^{\infty}\left(t^{-n}+t^{-n-2}\right)dt$

$\Rightarrow I=\frac{1}{2}\left[\frac{t^{-n+1}}{1-n}+\frac{t^{-n-1}}{-n-1}\right]_1^{\infty}=\frac{1}{2}\left[(0+0)-\left(\frac{1}{1-n}-\frac{1}{1+n}\right)\right]=\frac{1}{2}\left[(0+0)-\left(\frac{1}{1-n}-\frac{1}{1+n}\right)\right]=\frac{n}{n^2-1}$

Q25. Let $I=\int_{-1/2}^{1/2}\left\{[x]+\log\left(\frac{1+x}{1-x}\right)\right\}dx=\int_{-1/2}^{1/2}[x]dx+\int_{-1/2}^{1/2}\log\left(\frac{1+x}{1-x}\right)dx$

Note that the function in 2nd integral is odd function so,

$I=\int_{-1/2}^{0}[x]dx+\int_{0}^{1/2}[x]dx+0 \qquad \Rightarrow I=\int_{-1/2}^{0}-1dx+\int_{0}^{1/2}0dx=-\left[x\right]_{-1/2}^{0}+0=-\frac{1}{2}$.

Q26. Method 1 :

We have $x=\int_0^{y}\frac{dt}{\sqrt{1+4t^2}}=\int_0^{y}\frac{dt}{\sqrt{1+(2t)^2}} \qquad \Rightarrow x=\frac{1}{2}\left[\log\left|2t+\sqrt{1+4t^2}\right|\right]_0^{y}=\frac{1}{2}\log\left|2y+\sqrt{1+4y^2}\right|$

On differentiating w.r.t. y both sides, we get :

$\frac{dx}{dy}=\frac{1}{2}\left[\frac{1}{2y+\sqrt{1+4y^2}}\left(2+\frac{1}{2\sqrt{1+4y^2}}(0+8y)\right)\right] \qquad \Rightarrow \frac{dx}{dy}=\left[\frac{1}{4y+2\sqrt{1+4y^2}}\left(\frac{2\sqrt{1+4y^2}+4y}{\sqrt{1+4y^2}}\right)\right]$

$\Rightarrow \frac{dy}{dx}=\sqrt{1+4y^2} \qquad \Rightarrow \frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}\sqrt{1+4y^2}$

$\Rightarrow \frac{d^2y}{dx^2}=\frac{1}{2\sqrt{1+4y^2}}\left(8y\frac{dy}{dx}\right) \qquad \therefore \frac{d^2y}{dx^2}=\frac{1}{2\sqrt{1+4y^2}}\left(8y\sqrt{1+4y^2}\right)=4y$.

Method 2 :

We have $x=\int_0^{y}\frac{dt}{\sqrt{1+4t^2}} \Rightarrow \frac{dx}{dy}=\frac{1}{\sqrt{1+4y^2}}\times\frac{d}{dy}(y)$ [On differentiating w.r.t. y both sides

$\Rightarrow \frac{dx}{dy}=\frac{1}{\sqrt{1+4y^2}} \Rightarrow \frac{dy}{dx}=\sqrt{1+4y^2} \qquad \therefore \frac{d^2y}{dx^2}=\frac{1}{2}[1+4y^2]^{-1/2}\left(8y\times\frac{dy}{dx}\right)$

$\Rightarrow \frac{d^2y}{dx^2}=4y[1+4y^2]^{-1/2}\left(\sqrt{1+4y^2}\right) \qquad \therefore \frac{d^2y}{dx^2}=4y$.

Q27. We have $\int_0^{\pi}[x]dx=\int_0^{1}[x]dx+\int_1^{2}[x]dx+\int_2^{3}[x]dx+\int_3^{\pi}[x]dx=\int_0^{1}0\,dx+\int_1^{2}1\,dx+\int_2^{3}2\,dx+\int_3^{\pi}3\,dx=3(\pi-2)$

Q28. We have $\lim_{x\to 0}\frac{\int_0^{x^2}\cos t^2\,dt}{x\sin x}=\lim_{x\to 0}\frac{\cos(x^2)^2\frac{d}{dx}(x^2)-\cos(0)^2\frac{d}{dx}(0)}{\sin x+x\cos x}$ [Using L' Hospital's Rule

$$\Rightarrow \quad = \lim_{x\to 0}\frac{2x\cos(x^4)-0}{\sin x + x\cos x} = \lim_{x\to 0}\frac{2\cos x^4 + 2x(-4x^3\sin x^4)}{\cos x + \cos x - x\sin x}$$

$$\Rightarrow \quad = \lim_{x\to 0}\frac{2\cos x^4 - 8x^4\sin x^4}{2\cos x - x\sin x} = \frac{2\cos 0 - 0}{2\cos 0 - 0} = 1$$

Q29. Let $I = \int_{-m}^{m}\sqrt{\frac{m-x}{m+x}}\,dx = \int_{-m}^{m}\frac{m-x}{\sqrt{m^2-x^2}}\,dx \quad \Rightarrow I = \int_{-m}^{m}\frac{m}{\sqrt{m^2-x^2}}\,dx - \int_{-m}^{m}\frac{x}{\sqrt{m^2-x^2}}\,dx$

[Note that function in 1st integral is even function and, in 2nd integral function is odd.]

$$\therefore I = 2m\int_0^m \frac{1}{\sqrt{m^2-x^2}}\,dx - 0 = 2m\left[\sin^{-1}\frac{x}{m}\right]_0^m = 2m\left[\sin^{-1}1 - \sin^{-1}0\right] \qquad \therefore I = m\pi .$$

Q30. LHS : $\int_0^1 \tan^{-1}\left(\frac{1}{1-x+x^2}\right)dx = \int_0^1 \tan^{-1}\left(\frac{x+(1-x)}{1-x(1-x)}\right)dx$

$$\Rightarrow \quad = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}(1-x)\,dx = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}[1-(1-x)]\,dx = \int_0^1 \tan^{-1}x\,dx + \int_0^1 \tan^{-1}x\,dx$$

$\Rightarrow \quad = 2\int_0^1 \tan^{-1}x\,dx = \text{RHS}$. Hence Proved.

Now, $\int_0^1 \tan^{-1}\left(1-x+x^2\right)dx = \int_0^1\left[\frac{\pi}{2} - \cot^{-1}\left(1-x+x^2\right)\right]dx = \int_0^1 \frac{\pi}{2}\,dx - 2\int_0^1 \tan^{-1}x\,dx$ ...(i)

Consider $\int 2\tan^{-1}x\,dx = \tan^{-1}x\int 2\,dx - \int\left(\frac{d}{dx}\tan^{-1}x\int 2\,dx\right)dx$

$$\Rightarrow \quad = 2x\tan^{-1}x - \int\frac{2x}{1+x^2}\,dx = 2x\tan^{-1}x - \log\left|1+x^2\right|$$

$\therefore$ By (i), $\int_0^1 \tan^{-1}\left(1-x+x^2\right)dx = \frac{\pi}{2}\left[x\right]_0^1 - \left[2x\tan^{-1}x - \log\left|1+x^2\right|\right]_0^1$

$$\Rightarrow \quad = \frac{\pi}{2}\left[1-0\right] - \left[\left(2\tan^{-1}1 - \log 2\right) - 0\right] = \log 2$$

Hii, All!
I hope this texture may have proved beneficial for you.
While going through this material, if you noticed any error(s) or, something which doesn't make sense to you, please bring it in my notice through SMS or Call at +91-9650350480 or Email at **theopgupta@gmail.com**.

With lots of Love & Blessings!

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