Solutions Of Questions Of Continuity & Differentiability From NCERT Class XII

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Q01. Show that the function defined by $g(x) = x - [x]$ is discontinuous at all integral points. Here $[x]$ denotes the greatest integer less than or equal to x. [NCERT Ex-5.1 Q19

Sol. The given function $g(x)$ is defined at all integral points.
Let n be an integer. Then $g(n) = n - [n] = n - n = 0$.
LHL (at $x = n$): $\lim_{x\to n^-} g(x) = \lim_{x\to n^-}(x-[x]) = n-(n-1) = 1$.
RHL (at $x = n$): $\lim_{x\to n^+} g(x) = \lim_{x\to n^+}(x-[x]) = n-(n) = 0$.
Since LHL (at $x = n$) $\neq$ RHL (at $x = n$).
Therefore g is not continuous at $x = n$ i.e., $g(x)$ is discontinuous at all integral points.

Q02. Is the function defined by $f(x) = x^2 - \sin x + 5$ continuous at $x = \pi$? [NCERT Ex-5.1 Q20

Sol. We have $f(x) = x^2 - \sin x + 5$
$\therefore f(\pi) = (\pi)^2 - \sin\pi + 5 = 5 + \pi^2$ …(i)
LHL (at $x = \pi$): $\lim_{x\to\pi-0} f(x) = \lim_{x\to\pi-0}(x^2 - \sin x + 5)$ [Put $x = \pi - h$ so that as $x \to \pi$, $h \to 0$
$= \lim_{h\to 0}((\pi-h)^2 - \sin(\pi-h)+5)$
$= \lim_{h\to 0}((\pi-h)^2 - \sin h+5) = ((\pi-0)^2 - \sin 0 + 5) = \pi^2 + 5$ …(ii)
RHL (at $x = \pi$): $\lim_{x\to\pi+0} f(x) = \lim_{x\to\pi+0}(x^2 - \sin x + 5)$ [Put $x = \pi + h$ so that as $x \to \pi$, $h \to 0$
$= \lim_{h\to 0}((\pi+h)^2 - \sin(\pi+h)+5)$
$= \lim_{h\to 0}((\pi+h)^2 + \sin h+5) = ((\pi-0)^2 + \sin 0 + 5) = \pi^2 + 5$ …(iii)
By (i), (ii) & (iii), it is clearly evident that LHL (at $x = \pi$) = RHL (at $x = \pi$) = $f(\pi)$.
So, f is continuous at $x = \pi$.

Q03. Discuss the continuity of the following functions:
(a) $f(x) = \sin x + \cos x$ (b) $f(x) = \sin x - \cos x$
(c) $f(x) = \sin x \,.\, \cos x$ [NCERT Ex-5.1 Q21

Sol. Since we know that if g and h are two continuous functions then, $g + h$, $g - h$, $g.h$ are also continuous. …(A)
So, in order to prove all (a), (b) and (c), it will be sufficient to prove that $\sin x$ and $\cos x$ are continuous functions.
Let $g(x) = \sin x$. Let c be any real number.
Then $g(c) = \sin c$ …(i) [As $\sin x$ is defined for every real number]
LHL (at $x = c$): $\lim_{x\to c^-} g(x) = \lim_{x\to c^-}\sin x$ [Put $x = c - h$ so that as $x \to c$, $h \to 0$
$= \lim_{h\to 0}\sin(c-h) = \sin(c-0) = \sin c$ …(ii)
RHL (at $x = c$): $\lim_{x\to c^+} g(x) = \lim_{x\to c^+}\sin x$ [Put $x = c + h$ so that as $x \to c$, $h \to 0$
$= \lim_{h\to 0}\sin(c+h) = \sin(c+0) = \sin c$ …(iii)
By (i), (ii) & (iii), it is clearly evident that LHL (at $x = c$) = RHL (at $x = c$) = $g(c)$.
So, $\sin x$ is continuous at all real values of x.

Also, let $h(x) = \cos x$. Let c be any real number.
Then $h(c) = \cos c$ …(iv) [As $\cos x$ is defined for every real number]
LHL (at $x = c$): $\lim_{x\to c^-} h(x) = \lim_{x\to c^-}\cos x$ [Put $x = c - h$ so that as $x \to c$, $h \to 0$
$= \lim_{h\to 0}\cos(c-h) = \cos(c-0) = \cos c$ …(v)
RHL (at $x = c$): $\lim_{x\to c^+} h(x) = \lim_{x\to c^+}\cos x$ [Put $x = c + h$ so that as $x \to c$, $h \to 0$

$= \lim_{h\to 0}\cos(c+h) = \cos(c+0) = \cos c \quad \ldots(vi)$

By (iv), (v) & (vi), it is clearly evident that LHL (at $x = c$) = RHL (at $x = c$) = $h(c)$.
So, cos x is continuous at all real values of x.

Hence by using the statement (A) mentioned above, we can say:

(a) $f(x) = \sin x + \cos x$ is continuous for all real values of x.

(b) $f(x) = \sin x - \cos x$ is continuous for all real values of x.

(c) $f(x) = \sin x \,.\, \cos x$ is continuous for all real values of x.

Q04. Discuss continuity of cosine, cosecant, secant and cotangent functions. [NCERT Ex-5.1 Q22

Sol. Note that if f and g are continuous functions, then

(i) $\dfrac{f(x)}{g(x)},\ g(x)\neq 0$ is continuous.

(ii) $\dfrac{1}{g(x)},\ g(x)\neq 0$ is continuous.

(iii) $\dfrac{1}{f(x)},\ f(x)\neq 0$ is continuous.

Now, for the discussion over continuity of cosine function, please see the previous question.
Also, if cos x is continuous (see the solution of previous question) then sec x is continuous as well for all $x\in R-(2n-1)\dfrac{\pi}{2}, n\in Z$ as $\sec x = \dfrac{1}{\cos x}$, $\cos x\neq 0$ (since cos x = 0 at all $x=(2n-1)\dfrac{\pi}{2}, n\in Z$).

Again, if sin x is continuous (see the solution of previous question) then cosec x is continuous as well for all $x\in R-n\pi,\ n\in Z$ as $\text{cosec}\, x = \dfrac{1}{\sin x}$, $\sin x\neq 0$ (since sin x = 0 at all $x = n\pi,\ n\in Z$).

Finally, cot x is continuous for all $x\in R-n\pi,\ n\in Z$ as $\cot x = \dfrac{\cos x}{\sin x}$, $\sin x\neq 0$ (since sin x = 0 at all $x = n\pi,\ n\in Z$).

Q05. Determine if f defined by $f(x)=\begin{cases} x^2\sin\left(\dfrac{1}{x}\right), & \text{if } x\neq 0 \\ 0, & \text{if } x=0\end{cases}$ is a continuous function?

[NCERT Ex-5.1 Q22

Sol. LHL (at $x = 0$): $\lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} f(-h)$ [Put $x = 0-h$ so that as $x\to 0,\ h\to 0$

$= \lim_{h\to 0}(-h)^2\sin\left(\dfrac{1}{-h}\right)$

$= \lim_{h\to 0}[-h^2\sin\left(\dfrac{1}{h}\right)] = 0.$ $\quad [\because -1\le \sin\left(\dfrac{1}{x}\right)\le 1$

RHL (at $x = 0$): $\lim_{x\to 0^+} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} f(h)$ [Put $x = 0+h$ so that as $x\to 0,\ h\to 0$

$= \lim_{h\to 0}(h)^2\sin\left(\dfrac{1}{h}\right)$

$= \lim_{h\to 0} h^2\sin\left(\dfrac{1}{h}\right) = 0.$ $\quad [\because -1\le \sin\left(\dfrac{1}{x}\right)\le 1$

And $f(0) = 0$.
Since LHL (at $x = 0$) = RHL (at $x = 0$) = $f(0)$.
Therefore f is continuous at $x = 0$.

Q06. Show that the function $f(x) = \cos(x^2)$ is a continuous function. [NCERT Ex-5.1 Q31

Sol. The function f is defined for all real number and it can be expressed as the composition of two functions $f = g \circ h$, where $g(x) = \cos x$ and $h(x) = x^2$.

[$\because$ $(g \circ h)(x) = g(h(x)) = g(x^2) = \cos x^2 = f(x)$]

So, we need to prove that $g(x)$ and $h(x)$ are continuous functions.

We have $g(x) = \cos x$. Let c be any real number.

Then $g(c) = \cos c$ ...(i) [As $\cos x$ is defined for every real number]

LHL (at $x = c$): $\lim_{x \to c^-} g(x) = \lim_{x \to c^-} \cos x$ [Put $x = c - h$ so that as $x \to c$, $h \to 0$

$= \lim_{h \to 0} \cos(c - h) = \cos(c - 0) = \cos c$ ...(ii)

RHL (at $x = c$): $\lim_{x \to c^+} g(x) = \lim_{x \to c^+} \cos x$ [Put $x = c + h$ so that as $x \to c$, $h \to 0$

$= \lim_{h \to 0} \cos(c + h) = \cos(c + 0) = \cos c$ ...(iii)

By (i), (ii) & (iii), it is clearly evident that LHL (at $x = c$) = RHL (at $x = c$) = $f(c)$.

So, $g(x)$ is continuous at all real values of x. [Continuity of $\cos x$ has been discussed in Q03 as well.]

Also, we have $h(x) = x^2$.

Clearly, the function h is defined for every real number.

Let k be a real number, then $h(k) = k^2$.

And $\lim_{x \to k} h(x) = \lim_{x \to k} x^2 = k^2$.

$\because$ $\lim_{x \to k} h(x) = h(k)$ $\Rightarrow$ $h(x)$ is a continuous function.

Since it is known that for real valued functions g and h, such that $(g \circ h)$ is defined at c, if h is continuous at c and if g is continuous at $h(c)$, then $(g \circ h)$ is continuous at c.

Therefore $f(x) = (g \circ h)(x) = \cos(x^2)$ is continuous function.

Q07. Show that the function $f(x) = |\cos x|$ is a continuous function. [NCERT Ex-5.1 Q32

Sol. The function f is defined for all real number and it can be expressed as the composition of two functions $f = g \circ h$, where $g(x) = |x|$ and $h(x) = \cos x$.

[$\because$ $(g \circ h)(x) = g(h(x)) = g(\cos x) = |\cos x| = f(x)$]

So, we need to prove that $g(x)$ and $h(x)$ are continuous functions.

We have $g(x) = |x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x < 0 \end{cases}$

Let c be a real number.

<u>Case I</u>: If $c < 0$, then $g(c) = -c$ and $\lim_{x \to c} g(x) = \lim_{x \to c}(-x) = -c$

$\because$ $\lim_{x \to c} g(x) = g(c)$

$\therefore$ $g(x)$ is continuous at all points x, such that $x < 0$.

<u>Case II</u>: If $c > 0$, then $g(c) = c$ and $\lim_{x \to c} g(x) = \lim_{x \to c}(x) = c$

$\because$ $\lim_{x \to c} g(x) = g(c)$

$\therefore$ $g(x)$ is continuous at all points x, such that $x > 0$.

<u>Case III</u>: If $c = 0$, then $g(0) = 0$.

LHL (at $x = 0$): $\lim_{x \to 0^-} g(x) = \lim_{x \to 0^-}(-x) = -0 = 0$.

RHL (at $x = 0$): $\lim_{x \to 0^+} g(x) = \lim_{x \to 0^+}(x) = 0$.

Clearly LHL (at $x = 0$) = RHL (at $x = 0$) = $g(0)$.

$\therefore$ $g(x)$ is continuous at $x = 0$.

Also $h(x) = \cos x$. Let c be any real number.

Then $h(c) = \cos c$ ...(i) [As $\cos x$ is defined for every real number]

LHL (at $x = c$): $\lim_{x \to c^-} h(x) = \lim_{x \to c^-} \cos x$ [Put $x = c - h$ so that as $x \to c$, $h \to 0$

$= \lim_{h\to 0}\cos(c-h) = \cos(c-0) = \cos c$ ...(ii)

RHL (at $x = c$): $\lim_{x\to c^+} h(x) = \lim_{x\to c^+}\cos x$ [Put $x = c+h$ so that as $x \to c, h \to 0$

$= \lim_{h\to 0}\cos(c+h) = \cos(c+0) = \cos c$ ...(iii)

By (i), (ii) & (iii), it is clearly evident that LHL (at $x = c$) = RHL (at $x = c$) = $h(c)$.
So, $h(x)$ is continuous at all real values of x.

Since it is known that for real valued functions g and h, such that (g o h) is defined at c, if h is continuous at c and if g is continuous at $h(c)$, then (g o h) is continuous at c.

Therefore $f(x) = (g \text{ o } h)(x) = g(h(x)) = g(\cos x) = |\cos x|$ is continuous function.

Q08. Examine that sin $|x|$ is a continuous function. [NCERT Ex-5.1 Q33

Sol. The function f is defined for all real number and it can be expressed as the composition of two functions $f = g$ o h, where $g(x) = \sin x$ and $h(x) = |x|$.
[$\because$ $(g \text{ o } h)(x) = g(h(x)) = g(|x|) = \sin|x| = f(x)$]
So, we need to prove that $g(x)$ and $h(x)$ are continuous functions.

We have $g(x) = \sin x$. Let c be any real number.
Then $g(c) = \sin c$ ...(i) [As $\sin x$ is defined for every real number.
LHL (at $x = c$): $\lim_{x\to c^-} g(x) = \lim_{x\to c^-}\sin x$ [Put $x = c-h$ so that as $x \to c, h \to 0$

$= \lim_{h\to 0}\sin(c-h) = \sin(c-0) = \sin c$ ...(ii)

RHL (at $x = c$): $\lim_{x\to c^+} g(x) = \lim_{x\to c^+}\sin x$ [Put $x = c+h$ so that as $x \to c, h \to 0$

$= \lim_{h\to 0}\sin(c+h) = \sin(c+0) = \sin c$ ...(iii)

By (i), (ii) & (iii), it is clearly evident that LHL (at $x = c$) = RHL (at $x = c$) = $g(c)$.
So, $\sin x$ is continuous at all real values of x.

Also we have $h(x) = |x| = \begin{cases} x, \text{ if } x \ge 0 \\ -x, \text{ if } x < 0 \end{cases}$

Let c be a real number.

<u>Case I</u>: If $c < 0$, then $h(c) = -c$ and $\lim_{x\to c} h(x) = \lim_{x\to c}(-x) = -c$

$\because \lim_{x\to c} h(x) = h(c)$

$\therefore$ $h(x)$ is continuous at all points x, such that $x < 0$.

<u>Case II</u>: If $c > 0$, then $h(c) = c$ and $\lim_{x\to c} h(x) = \lim_{x\to c}(x) = c$

$\because \lim_{x\to c} h(x) = h(c)$

$\therefore$ $h(x)$ is continuous at all points x, such that $x > 0$.

<u>Case III</u>: If $c = 0$, then $h(0) = 0$.
LHL (at $x = 0$): $\lim_{x\to 0^-} h(x) = \lim_{x\to 0^-}(-x) = -0 = 0$.
RHL (at $x = 0$): $\lim_{x\to 0^+} h(x) = \lim_{x\to 0^+}(x) = 0$.
Clearly LHL (at $x = 0$) = RHL (at $x = 0$) = $h(0)$.
$\therefore$ $h(x)$ is continuous at $x = 0$.

Since it is known that for real valued functions g and h, such that (g o h) is defined at c, if g is continuous at c and if f is continuous at $g(c)$, then (f o g) is continuous at c.

Therefore $f(x) = (g \text{ o } h)(x) = g(h(x)) = g(|x|) = \sin|x|$ is continuous function.

Q09. Find all the points of discontinuity of f defined by $f(x) = |x| - |x+1|$. [NCERT Ex-5.1 Q31

Sol. We have $f(x) = |x| - |x+1|$ i.e., $f(x) = g(x) + h(x)$ where, $g(x) = |x|$ and $h(x) = |x+1|$.

We shall firstly, examine the continuity of $g(x)$ and $h(x)$.

We have $g(x)=|x|=\begin{cases} x, \text{ if } x\ge 0 \\ -x, \text{ if } x<0 \end{cases}$

Let c be a real number.

Case I: If $c < 0$, then $g(c) = -c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(-x)=-c$

$\because \lim\limits_{x\to c} g(x)=g(c)$

$\therefore$ $g(x)$ is continuous at all points x, such that $x < 0$.

Case II: If $c > 0$, then $g(c) = c$ and $\lim\limits_{x\to c} g(x)=\lim\limits_{x\to c}(x)=c$

$\because \lim\limits_{x\to c} g(x)=g(c)$

$\therefore$ $g(x)$ is continuous at all points x, such that $x > 0$.

Case III: If $c = 0$, then $g(0) = 0$.

LHL (at $x = 0$): $\lim\limits_{x\to 0^-} g(x)=\lim\limits_{x\to 0^-}(-x)=-0=0$.

RHL (at $x = 0$): $\lim\limits_{x\to 0+} g(x)=\lim\limits_{x\to 0^+}(x)=0$.

Clearly LHL (at $x = 0$) = RHL (at $x = 0$) = $g(0)$.

$\therefore$ $g(x)$ is continuous at $x = 0$.

Now $h(x)=|x+1|=\begin{cases} x+1, \text{ if } x\ge -1 \\ -(x+1), \text{ if } x<-1 \end{cases}$

Let c be a real number.

Case I: If $c < -1$, then $h(c) = -(c + 1)$ and $\lim\limits_{x\to c} h(x)=\lim\limits_{x\to c}[-(x+1)]=-(c+1)$

$\because \lim\limits_{x\to c} h(x)=h(c)$

$\therefore$ $h(x)$ is continuous at all points x, such that $x < -1$.

Case II: If $c > -1$, then $h(c) = c + 1$ and $\lim\limits_{x\to c} h(x)=\lim\limits_{x\to c}(x+1)=c+1$

$\because \lim\limits_{x\to c} h(x)=h(c)$

$\therefore$ $h(x)$ is continuous at all points x, such that $x > -1$.

Case III: If $c = -1$, then $h(-1) = -1 + 1 = 0$.

LHL (at $x = -1$): $\lim\limits_{x\to -1^-} h(x)=\lim\limits_{x\to -1^-}(-x-1)=-(-1)-1=0$.

RHL (at $x = -1$): $\lim\limits_{x\to -1^+} h(x)=\lim\limits_{x\to 0^+}(x+1)=-1+1=0$.

Clearly LHL (at $x = -1$) = RHL (at $x = -1$) = $h(-1)$.

$\therefore$ $h(x)$ is continuous at $x = -1$.

Since g and h are continuous functions, therefore $f(x) = g(x) + h(x)$ is also continuous.
Hence, there is no point of discontinuity for the function $f(x)$.

Q10. Show that the function f defined by $f(x) = |1 - x + |x||$, where x is any real number, is a continuous function. [NCERT Example 20

Sol. Let $g(x) = 1 - x + |x|$ and $h(x) = |x|$ for all real x.

Then, $(h \circ g)(x) = h(g(x)) = h(1 - x + |x|) = |1 - x + |x|| = f(x)$.

We know that $|x|$ is a continuous function. Hence $h(x)$ is continuous function. So, $g(x)$ being the sum of a polynomial function and the modulus function is also continuous.

Finally the function f being a composite of two continuous functions is continuous as well.

Q11. Find all the points of discontinuity of the greatest integer function defined by $f(x) = [x]$, where $[x]$ denotes the greatest integer less than or equal to x. [NCERT Example 15

Sol. We have $f(x) = [x]$.

Case I: Let c be a real number which is not equal to any integer. It is evident that for all real numbers close to c the value of the function is equal to $[c]$; i.e., $\lim\limits_{x\to c} f(x)=\lim\limits_{x\to c}[x]=[c]$.

Also $f(c) = [c]$ and hence the greatest integer function is continuous at all non-integral real numbers (since $\lim\limits_{x\to c} f(x)=f(c)=[c]$).

Case II: Let c be an integer.

Then $\lim_{x\to c^-}[x]=c-1$ and $\lim_{x\to c^+}[x]=c$.

Since $\lim_{x\to c^-} f(x)\neq \lim_{x\to c^+} f(x)$, therefore the greatest integer function is discontinuous at all integral points.

Q12. Prove that $f(x) = |x-1|$, $x \in R$ is not differentiable at $x = 1$. [NCERT Ex-5.2 Q09

Sol. We have $f(x)=|x-1|=\begin{cases} x-1, \text{ if } x\geq 1 \\ -(x-1), \text{ if } x<1 \end{cases}$.

Then $f(1) = 1 - 1 = 0$.

We know that a function is differentiable at a point $x = m$ in its domain if LHD and RHD at $x = m$ are both finite and equal to each other.

Differentiability at $x = 1$:

LHD (at $x = 1$): $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}$

$$=\lim_{x\to 1^-}\frac{[-(x-1)]-(0)}{x-1}=\lim_{x\to 1^-}\frac{-(x-1)}{x-1}=\lim_{x\to 1^-}(-1)=-1.$$

RHD (at $x = 1$): $\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$

$$=\lim_{x\to 1^+}\frac{(x-1)-(0)}{x-1}=\lim_{x\to 1^-}\frac{(x-1)}{x-1}=\lim_{x\to 1^-}(1)=1.$$

Since LHD (at $x = 1$) $\neq$ RHD (at $x = 1$) so, f is not differentiable at $x = 1$.

Q13. Prove that the greatest integer function defined by $f(x) = [x]$, $0 < x < 3$ is not differentiable at $x = 1$ and $x = 2$. [NCERT Ex-5.2 Q10

Sol. We have $f(x) = [x]$, $0 < x < 3$.

As a function is differentiable at a point $x = m$ in its domain if LHD and RHD at $x = m$ are both finite and equal to each other.

Differentiability at $x = 1$:

LHD (at $x = 1$): $\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1}=\lim_{h\to 0}\frac{f(1-h)-f(1)}{(1-h)-1}$ [Put $x=1-h$ so that as $x\to 1$, $h\to 0$

$$=\lim_{h\to 0}\frac{[1-h]-[1]}{-h}=\lim_{h\to 0}\frac{0-1}{-h}=\lim_{h\to 0}\frac{1}{h}=\frac{1}{0}=\infty.$$

RHD (at $x = 1$): $\lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}=\lim_{h\to 0}\frac{f(1+h)-f(1)}{(1+h)-1}$ [Put $x=1+h$ so that as $x\to 1$, $h\to 0$

$$=\lim_{h\to 0}\frac{[1+h]-[1]}{h}=\lim_{h\to 0}\frac{1-1}{h}=\lim_{h\to 0}\frac{0}{h}=\lim_{h\to 0}(0)=0.$$

Since LHD (at $x = 1$) $\neq$ RHD (at $x = 1$) so, f is not differentiable at $x = 1$.

[**NOTE** that here we didn't need to evaluate RHD as LHD is already not defined (i.e., ∞), which is an enough reason for f to not be differentiable at $x = 1$.]

Differentiability at $x = 2$:

LHD (at $x = 2$): $\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}$

$$=\lim_{x\to 2^-}\frac{[x]-[2]}{x-2}=\lim_{x\to 2^-}\frac{1-2}{x-2}=\lim_{x\to 2^-}\frac{-1}{x-2}=\frac{-1}{0}=-\infty.$$

Since LHD (at $x = 2$) doesn't exist so, f is not differentiable at $x = 2$.