

Exercise 4.1**Question 1:**

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{(3^n - 1)}{2}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1 + 3 + 3^2 + \dots + 3^{n-1} = \frac{(3^n - 1)}{2}$$

For $n = 1$, we have

$$P(1): 1 = \frac{(3^1 - 1)}{2} = \frac{3-1}{2} = \frac{2}{2} = 1, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1 + 3 + 3^2 + \dots + 3^{k-1} = \frac{(3^k - 1)}{2} \quad \dots(i)$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & 1 + 3 + 3^2 + \dots + 3^{k-1} + 3^{(k+1)-1} \\ &= (1 + 3 + 3^2 + \dots + 3^{k-1}) + 3^k \end{aligned}$$

$$= \frac{(3^k - 1)}{2} + 3^k \quad \text{[Using (i)]}$$

$$= \frac{(3^k - 1) + 2 \cdot 3^k}{2}$$

$$= \frac{(1+2)3^k - 1}{2}$$

$$= \frac{3 \cdot 3^k - 1}{2}$$

$$= \frac{3^{k+1} - 1}{2}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 2:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1^3 + 2^3 + 3^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

For $n = 1$, we have

$$P(1): 1^3 = 1 = \left(\frac{1(1+1)}{2} \right)^2 = \left(\frac{1 \cdot 2}{2} \right)^2 = 1^2 = 1, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \left(\frac{k(k+1)}{2} \right)^2 \quad \dots \text{ (i)}$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$1^3 + 2^3 + 3^3 + \dots + k^3 + (k + 1)^3$$

$$\begin{aligned}
&= \left(\frac{k(k+1)}{2} \right)^2 + (k+1)^3 && \text{[Using (i)]} \\
&= \frac{k^2(k+1)^2}{4} + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{(k+1)^2 \{k^2 + 4(k+1)\}}{4} \\
&= \frac{(k+1)^2 \{k^2 + 4k + 4\}}{4} \\
&= \frac{(k+1)^2 (k+2)^2}{4} \\
&= \frac{(k+1)^2 (k+1+1)^2}{4} \\
&= (1^3 + 2^3 + 3^3 + \dots + k^3) + (k+1)^3 = \left(\frac{(k+1)(k+1+1)}{2} \right)^2
\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 3:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1 + \frac{1}{(1+2)} + \frac{1}{(1+2+3)} + \dots + \frac{1}{(1+2+3+\dots+n)} = \frac{2n}{(n+1)}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+n} = \frac{2n}{n+1}$$

For $n = 1$, we have

$$P(1): 1 = \frac{2 \cdot 1}{1+1} = \frac{2}{2} = 1 \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1 + \frac{1}{1+2} + \dots + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k} = \frac{2k}{k+1} \quad \dots (i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\begin{aligned} & 1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k} + \frac{1}{1+2+3+\dots+k+(k+1)} \\ &= \left(1 + \frac{1}{1+2} + \frac{1}{1+2+3} + \dots + \frac{1}{1+2+3+\dots+k} \right) + \frac{1}{1+2+3+\dots+k+(k+1)} \\ &= \frac{2k}{k+1} + \frac{1}{1+2+3+\dots+k+(k+1)} \quad \text{[Using (i)]} \\ &= \frac{2k}{k+1} + \frac{1}{\left(\frac{(k+1)(k+1+1)}{2} \right)} \quad \left[1+2+3+\dots+n = \frac{n(n+1)}{2} \right] \\ &= \frac{2k}{(k+1)} + \frac{2}{(k+1)(k+2)} \\ &= \frac{2}{(k+1)} \left(k + \frac{1}{k+2} \right) \end{aligned}$$

Question 4:

Prove the following by using the principle of mathematical induction for all $n \in N$: 1.2.3

$$+ 2.3.4 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1.2.3 + 2.3.4 + \dots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$$

For $n = 1$, we have

$$P(1): 1.2.3 = 6 = \frac{1(1+1)(1+2)(1+3)}{4} = \frac{1.2.3.4}{4} = 6, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1.2.3 + 2.3.4 + \dots + k(k+1)(k+2) = \frac{k(k+1)(k+2)(k+3)}{4} \quad \dots (i)$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$1.2.3 + 2.3.4 + \dots + k(k + 1)(k + 2) + (k + 1)(k + 2)(k + 3) \\ = \{1.2.3 + 2.3.4 + \dots + k(k + 1)(k + 2)\} + (k + 1)(k + 2)(k + 3)$$

$$= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \quad [\text{Using (i)}]$$

$$= (k+1)(k+2)(k+3) \left(\frac{k}{4} + 1 \right)$$

$$= \frac{(k+1)(k+2)(k+3)(k+4)}{4}$$

$$= \frac{(k+1)(k+1+1)(k+1+2)(k+1+3)}{4}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 5:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n) : 1.3 + 2.3^2 + 3.3^3 + \dots + n.3^n = \frac{(2n-1)3^{n+1} + 3}{4}$$

For $n = 1$, we have

$$P(1): 1.3 = 3 = \frac{(2.1-1)3^{1+1} + 3}{4} = \frac{3^2 + 3}{4} = \frac{12}{4} = 3, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k = \frac{(2k-1)3^{k+1} + 3}{4} \quad \dots \text{ (i)}$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned}
 & 1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k + (k+1)3^{k+1} \\
 &= (1.3 + 2.3^2 + 3.3^3 + \dots + k.3^k) + (k+1)3^{k+1} \\
 &= \frac{(2k-1)3^{k+1} + 3}{4} + (k+1)3^{k+1} \quad \text{[Using (i)]} \\
 &= \frac{(2k-1)3^{k+1} + 3 + 4(k+1)3^{k+1}}{4} \\
 &= \frac{3^{k+1}\{2k-1+4(k+1)\} + 3}{4} \\
 &= \frac{3^{k+1}\{6k+3\} + 3}{4} \\
 &= \frac{3^{k+1}.3\{2k+1\} + 3}{4} \\
 &= \frac{3^{(k+1)+1}\{2k+1\} + 3}{4} \\
 &= \frac{\{2(k+1)-1\}3^{(k+1)+1} + 3}{4}
 \end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 6:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1.2 + 2.3 + 3.4 + \dots + n.(n+1) = \left[\frac{n(n+1)(n+2)}{3} \right]$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1.2 + 2.3 + 3.4 + \dots + n.(n+1) = \left[\frac{n(n+1)(n+2)}{3} \right]$$

For $n = 1$, we have

$$P(1): 1.2 = 2 = \frac{1(1+1)(1+2)}{3} = \frac{1.2.3}{3} = 2, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1.2 + 2.3 + 3.4 + \dots + k.(k+1) = \left[\frac{k(k+1)(k+2)}{3} \right] \quad \dots (i)$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & 1.2 + 2.3 + 3.4 + \dots + k.(k+1) + (k+1).(k+2) \\ &= [1.2 + 2.3 + 3.4 + \dots + k.(k+1)] + (k+1).(k+2) \\ &= \frac{k(k+1)(k+2)}{3} + (k+1)(k+2) \quad \text{[Using (i)]} \\ &= (k+1)(k+2) \left(\frac{k}{3} + 1 \right) \\ &= \frac{(k+1)(k+2)(k+3)}{3} \\ &= \frac{(k+1)(k+1+1)(k+1+2)}{3} \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 7:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1.3 + 3.5 + 5.7 + \dots + (2n-1)(2n+1) = \frac{n(4n^2 + 6n - 1)}{3}$$

For $n = 1$, we have

$$P(1): 1.3 = 3 = \frac{1(4.1^2 + 6.1 - 1)}{3} = \frac{4 + 6 - 1}{3} = \frac{9}{3} = 3, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1) = \frac{k(4k^2 + 6k - 1)}{3} \quad \dots \text{(i)}$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$(1.3 + 3.5 + 5.7 + \dots + (2k-1)(2k+1) + \{2(k+1)-1\}\{2(k+1)+1\})$$

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k+2-1)(2k+2+1) \quad [\text{Using (i)}]$$

$$= \frac{k(4k^2 + 6k - 1)}{3} + (2k+1)(2k+3)$$

$$= \frac{k(4k^2 + 6k - 1)}{3} + (4k^2 + 8k + 3)$$

$$= \frac{k(4k^2 + 6k - 1) + 3(4k^2 + 8k + 3)}{3}$$

$$= \frac{4k^3 + 6k^2 - k + 12k^2 + 24k + 9}{3}$$

$$= \frac{4k^3 + 18k^2 + 23k + 9}{3}$$

$$= \frac{4k^3 + 14k^2 + 9k + 4k^2 + 14k + 9}{3}$$

$$= \frac{k(4k^2 + 14k + 9) + 1(4k^2 + 14k + 9)}{3}$$

$$= \frac{(k+1)(4k^2 + 14k + 9)}{3}$$

$$= \frac{(k+1)\{4k^2 + 8k + 4 + 6k + 6 - 1\}}{3}$$

$$= \frac{(k+1)\{4(k^2 + 2k + 1) + 6(k+1) - 1\}}{3}$$

$$= \frac{(k+1)\{4(k+1)^2 + 6(k+1) - 1\}}{3}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 8:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$: $1.2 + 2.2^2 + 3.2^2 + \dots + n.2^n = (n - 1) 2^{n+1} + 2$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1.2 + 2.2^2 + 3.2^2 + \dots + n.2^n = (n - 1) 2^{n+1} + 2$$

For $n = 1$, we have

$$P(1): 1.2 = 2 = (1 - 1) 2^{1+1} + 2 = 0 + 2 = 2, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$1.2 + 2.2^2 + 3.2^2 + \dots + k.2^k = (k - 1) 2^{k+1} + 2 \dots \text{ (i)}$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & \{1.2 + 2.2^2 + 3.2^3 + \dots + k.2^k\} + (k+1).2^{k+1} \\ &= (k-1)2^{k+1} + 2 + (k+1)2^{k+1} \\ &= 2^{k+1} \{(k-1) + (k+1)\} + 2 \\ &= 2^{k+1}.2k + 2 \\ &= k.2^{(k+1)+1} + 2 \\ &= \{(k+1) - 1\} 2^{(k+1)+1} + 2 \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 9:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

For $n = 1$, we have

$$P(1): \frac{1}{2} = 1 - \frac{1}{2^1} = \frac{1}{2}, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} = 1 - \frac{1}{2^k} \quad \dots \text{ (i)}$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^k} \right) + \frac{1}{2^{k+1}} \\ &= \left(1 - \frac{1}{2^k} \right) + \frac{1}{2^{k+1}} \quad \text{[Using (i)]} \\ &= 1 - \frac{1}{2^k} + \frac{1}{2 \cdot 2^k} \\ &= 1 - \frac{1}{2^k} \left(1 - \frac{1}{2} \right) \\ &= 1 - \frac{1}{2^k} \left(\frac{1}{2} \right) \\ &= 1 - \frac{1}{2^{k+1}} \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 10:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3n-1)(3n+2)} = \frac{n}{(6n+4)}$$

For $n = 1$, we have

$$P(1) = \frac{1}{2.5} = \frac{1}{10} = \frac{1}{6.1+4} = \frac{1}{10}, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$\frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} = \frac{k}{6k+4} \quad \dots (i)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\begin{aligned} & \frac{1}{2.5} + \frac{1}{5.8} + \frac{1}{8.11} + \dots + \frac{1}{(3k-1)(3k+2)} + \frac{1}{\{3(k+1)-1\}\{3(k+1)+2\}} \\ &= \frac{k}{6k+4} + \frac{1}{(3k+3-1)(3k+3+2)} \quad \text{[Using (i)]} \\ &= \frac{k}{6k+4} + \frac{1}{(3k+2)(3k+5)} \\ &= \frac{k}{2(3k+2)} + \frac{1}{(3k+2)(3k+5)} \\ &= \frac{1}{(3k+2)} \left(\frac{k}{2} + \frac{1}{3k+5} \right) \\ &= \frac{1}{(3k+2)} \left(\frac{k(3k+5)+2}{2(3k+5)} \right) \\ &= \frac{1}{(3k+2)} \left(\frac{3k^2+5k+2}{2(3k+5)} \right) \\ &= \frac{1}{(3k+2)} \left(\frac{(3k+2)(k+1)}{2(3k+5)} \right) \\ &= \frac{(k+1)}{6k+10} \\ &= \frac{(k+1)}{6(k+1)+4} \end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 11:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$\frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): \frac{1}{1.2.3} + \frac{1}{2.3.4} + \frac{1}{3.4.5} + \dots + \frac{1}{n(n+1)(n+2)} = \frac{n(n+3)}{4(n+1)(n+2)}$$

For $n = 1$, we have

$$P(1): \frac{1}{1 \cdot 2 \cdot 3} = \frac{1 \cdot (1+3)}{4(1+1)(1+2)} = \frac{1 \cdot 4}{4 \cdot 2 \cdot 3} = \frac{1}{1 \cdot 2 \cdot 3}, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{k(k+1)(k+2)} = \frac{k(k+3)}{4(k+1)(k+2)} \quad \dots \text{ (i)}$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned}
& \left[\frac{1}{1 \cdot 2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \frac{1}{3 \cdot 4 \cdot 5} + \dots + \frac{1}{k(k+1)(k+2)} \right] + \frac{1}{(k+1)(k+2)(k+3)} \\
&= \frac{k(k+3)}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)} \quad \text{[Using (i)]} \\
&= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+3)}{4} + \frac{1}{k+3} \right\} \\
&= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+3)^2 + 4}{4(k+3)} \right\} \\
&= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k^2 + 6k + 9) + 4}{4(k+3)} \right\} \\
&= \frac{1}{(k+1)(k+2)} \left\{ \frac{k^3 + 6k^2 + 9k + 4}{4(k+3)} \right\} \\
&= \frac{1}{(k+1)(k+2)} \left\{ \frac{k^3 + 2k^2 + k + 4k^2 + 8k + 4}{4(k+3)} \right\} \\
&= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k^2 + 2k + 1) + 4(k^2 + 2k + 1)}{4(k+3)} \right\} \\
&= \frac{1}{(k+1)(k+2)} \left\{ \frac{k(k+1)^2 + 4(k+1)^2}{4(k+3)} \right\} \\
&= \frac{(k+1)^2(k+4)}{4(k+1)(k+2)(k+3)} \\
&= \frac{(k+1)\{(k+1)+3\}}{4\{(k+1)+1\}\{(k+1)+2\}}
\end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 12:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): a + ar + ar^2 + \dots + ar^{n-1} = \frac{a(r^n - 1)}{r - 1}$$

For $n = 1$, we have

$$P(1): a = \frac{a(r^1 - 1)}{(r - 1)}, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$a + ar + ar^2 + \dots + ar^{k-1} = \frac{a(r^k - 1)}{r - 1} \quad \dots \text{ (i)}$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & \{a + ar + ar^2 + \dots + ar^{k-1}\} + ar^{(k+1)-1} \\ &= \frac{a(r^k - 1)}{r - 1} + ar^k \quad \quad \quad [\text{Using (i)}] \\ &= \frac{a(r^k - 1) + ar^k(r - 1)}{r - 1} \\ &= \frac{a(r^k - 1) + ar^{k+1} - ar^k}{r - 1} \\ &= \frac{ar^k - a + ar^{k+1} - ar^k}{r - 1} \\ &= \frac{ar^{k+1} - a}{r - 1} \\ &= \frac{a(r^{k+1} - 1)}{r - 1} \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 13:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\dots\left(1 + \frac{(2n+1)}{n^2}\right) = (n+1)^2$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): \left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\dots\left(1 + \frac{(2n+1)}{n^2}\right) = (n+1)^2$$

For $n = 1$, we have

$$P(1): \left(1 + \frac{3}{1}\right) = 4 = (1+1)^2 = 2^2 = 4, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\dots\left(1 + \frac{(2k+1)}{k^2}\right) = (k+1)^2 \quad \dots (1)$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & \left[\left(1 + \frac{3}{1}\right)\left(1 + \frac{5}{4}\right)\left(1 + \frac{7}{9}\right)\dots\left(1 + \frac{(2k+1)}{k^2}\right) \right] \left[1 + \frac{\{2(k+1)+1\}}{(k+1)^2} \right] \\ &= (k+1)^2 \left(1 + \frac{2(k+1)+1}{(k+1)^2} \right) \quad [\text{Using (1)}] \\ &= (k+1)^2 \left[\frac{(k+1)^2 + 2(k+1)+1}{(k+1)^2} \right] \\ &= (k+1)^2 + 2(k+1)+1 \\ &= \{(k+1)+1\}^2 \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 14:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\dots\left(1 + \frac{1}{n}\right) = (n+1)$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\dots\left(1 + \frac{1}{n}\right) = (n+1)$$

For $n = 1$, we have

$$P(1): \left(1 + \frac{1}{1}\right) = 2 = (1+1), \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$P(k): \left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\dots\left(1 + \frac{1}{k}\right) = (k+1) \quad \dots (1)$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & \left[\left(1 + \frac{1}{1}\right)\left(1 + \frac{1}{2}\right)\left(1 + \frac{1}{3}\right)\dots\left(1 + \frac{1}{k}\right) \right] \left(1 + \frac{1}{k+1}\right) \\ &= (k+1) \left(1 + \frac{1}{k+1}\right) \quad \text{[Using (1)]} \\ &= (k+1) \left(\frac{(k+1)+1}{(k+1)} \right) \\ &= (k+1)+1 \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 15:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n) = 1^2 + 3^2 + 5^2 + \dots + (2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$

For $n = 1$, we have

$$P(1) = 1^2 = 1 = \frac{1(2 \cdot 1 - 1)(2 \cdot 1 + 1)}{3} = \frac{1 \cdot 1 \cdot 3}{3} = 1, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$P(k) = 1^2 + 3^2 + 5^2 + \dots + (2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \dots (1)$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & \{1^2 + 3^2 + 5^2 + \dots + (2k-1)^2\} + \{2(k+1)-1\}^2 \\ &= \frac{k(2k-1)(2k+1)}{3} + (2k+2-1)^2 && \text{[Using (1)]} \\ &= \frac{k(2k-1)(2k+1)}{3} + (2k+1)^2 \\ &= \frac{k(2k-1)(2k+1) + 3(2k+1)^2}{3} \\ &= \frac{(2k+1)\{k(2k-1) + 3(2k+1)\}}{3} \\ &= \frac{(2k+1)\{2k^2 - k + 6k + 3\}}{3} \end{aligned}$$

$$\begin{aligned}
&= \frac{(2k+1)\{2k^2+5k+3\}}{3} \\
&= \frac{(2k+1)\{2k^2+2k+3k+3\}}{3} \\
&= \frac{(2k+1)\{2k(k+1)+3(k+1)\}}{3} \\
&= \frac{(2k+1)(k+1)(2k+3)}{3} \\
&= \frac{(k+1)\{2(k+1)-1\}\{2(k+1)+1\}}{3}
\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 16:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$\frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3n-2)(3n+1)} = \frac{n}{(3n+1)}$$

For $n=1$, we have

$$P(1) = \frac{1}{1.4} = \frac{1}{3.1+1} = \frac{1}{4} = \frac{1}{1.4}, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$P(k) = \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} = \frac{k}{3k+1} \quad \dots (1)$$

We shall now prove that $P(k+1)$ is true.

Consider

$$\begin{aligned}
& \left\{ \frac{1}{1.4} + \frac{1}{4.7} + \frac{1}{7.10} + \dots + \frac{1}{(3k-2)(3k+1)} \right\} + \frac{1}{\{3(k+1)-2\}\{3(k+1)+1\}} \\
&= \frac{k}{3k+1} + \frac{1}{(3k+1)(3k+4)} \quad [\text{Using (1)}] \\
&= \frac{1}{(3k+1)} \left\{ k + \frac{1}{(3k+4)} \right\} \\
&= \frac{1}{(3k+1)} \left\{ \frac{k(3k+4)+1}{(3k+4)} \right\} \\
&= \frac{1}{(3k+1)} \left\{ \frac{3k^2+4k+1}{(3k+4)} \right\} \\
&= \frac{1}{(3k+1)} \left\{ \frac{3k^2+3k+k+1}{(3k+4)} \right\} \\
&= \frac{(3k+1)(k+1)}{(3k+1)(3k+4)} \\
&= \frac{(k+1)}{3(k+1)+1}
\end{aligned}$$

Thus, $P(k+1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 17:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2n+1)(2n+3)} = \frac{n}{3(2n+3)}$$

For $n = 1$, we have

$$P(1): \frac{1}{3.5} = \frac{1}{3(2.1+3)} = \frac{1}{3.5}, \text{ which is true.}$$

Let $P(k)$ be true for some positive integer k , i.e.,

$$P(k): \frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} = \frac{k}{3(2k+3)} \quad \dots (1)$$

We shall now prove that $P(k + 1)$ is true.

Consider

$$\begin{aligned} & \left[\frac{1}{3.5} + \frac{1}{5.7} + \frac{1}{7.9} + \dots + \frac{1}{(2k+1)(2k+3)} \right] + \frac{1}{\{2(k+1)+1\}\{2(k+1)+3\}} \\ &= \frac{k}{3(2k+3)} + \frac{1}{(2k+3)(2k+5)} \quad [\text{Using (1)}] \\ &= \frac{1}{(2k+3)} \left[\frac{k}{3} + \frac{1}{(2k+5)} \right] \\ &= \frac{1}{(2k+3)} \left[\frac{k(2k+5)+3}{3(2k+5)} \right] \\ &= \frac{1}{(2k+3)} \left[\frac{2k^2+5k+3}{3(2k+5)} \right] \\ &= \frac{1}{(2k+3)} \left[\frac{2k^2+2k+3k+3}{3(2k+5)} \right] \\ &= \frac{1}{(2k+3)} \left[\frac{2k(k+1)+3(k+1)}{3(2k+5)} \right] \\ &= \frac{(k+1)(2k+3)}{3(2k+3)(2k+5)} \\ &= \frac{(k+1)}{3\{2(k+1)+3\}} \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 18:

Prove the following by using the principle of mathematical induction for all $n \in \mathbb{N}$:

$$1 + 2 + 3 + \dots + n < \frac{1}{8}(2n+1)^2$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 1 + 2 + 3 + \dots + n < \frac{1}{8}(2n+1)^2$$

It can be noted that $P(n)$ is true for $n = 1$ since $1 < \frac{1}{8}(2 \cdot 1 + 1)^2 = \frac{9}{8}$.

Let $P(k)$ be true for some positive integer k , i.e.,

$$1 + 2 + \dots + k < \frac{1}{8}(2k+1)^2 \quad \dots (1)$$

We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true.

Consider

$$\begin{aligned} (1 + 2 + \dots + k) + (k+1) &< \frac{1}{8}(2k+1)^2 + (k+1) && [\text{Using (1)}] \\ &< \frac{1}{8}\{(2k+1)^2 + 8(k+1)\} \\ &< \frac{1}{8}\{4k^2 + 4k + 1 + 8k + 8\} \\ &< \frac{1}{8}\{4k^2 + 12k + 9\} \\ &< \frac{1}{8}(2k+3)^2 \\ &< \frac{1}{8}\{2(k+1)+1\}^2 \end{aligned}$$

Hence, $(1 + 2 + 3 + \dots + k) + (k+1) < \frac{1}{8}(2k+1)^2 + (k+1)$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 19:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$: $n(n + 1)(n + 5)$ is a multiple of 3.

Answer

Let the given statement be $P(n)$, i.e.,

$P(n)$: $n(n + 1)(n + 5)$, which is a multiple of 3.

It can be noted that $P(n)$ is true for $n = 1$ since $1(1 + 1)(1 + 5) = 12$, which is a multiple of 3.

Let $P(k)$ be true for some positive integer k , i.e.,

$k(k + 1)(k + 5)$ is a multiple of 3.

$\therefore k(k + 1)(k + 5) = 3m$, where $m \in \mathbf{N} \dots (1)$

We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true.

Consider

$$\begin{aligned} & (k+1)\{(k+1)+1\}\{(k+1)+5\} \\ &= (k+1)(k+2)\{(k+5)+1\} \\ &= (k+1)(k+2)(k+5) + (k+1)(k+2) \\ &= \{k(k+1)(k+5) + 2(k+1)(k+5)\} + (k+1)(k+2) \\ &= 3m + (k+1)\{2(k+5) + (k+2)\} \\ &= 3m + (k+1)\{2k+10+k+2\} \\ &= 3m + (k+1)(3k+12) \\ &= 3m + 3(k+1)(k+4) \\ &= 3\{m + (k+1)(k+4)\} = 3 \times q, \text{ where } q = \{m + (k+1)(k+4)\} \text{ is some natural number} \end{aligned}$$

Therefore, $(k+1)\{(k+1)+1\}\{(k+1)+5\}$ is a multiple of 3.

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 20:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$: $10^{2n-1} + 1$ is divisible by 11.

Answer

Let the given statement be $P(n)$, i.e.,

$P(n)$: $10^{2^n - 1} + 1$ is divisible by 11.

It can be observed that $P(n)$ is true for $n = 1$ since $P(1) = 10^{2^1 - 1} + 1 = 11$, which is divisible by 11.

Let $P(k)$ be true for some positive integer k , i.e.,

$10^{2^k - 1} + 1$ is divisible by 11.

$\therefore 10^{2^k - 1} + 1 = 11m$, where $m \in \mathbf{N} \dots (1)$

We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true.

Consider

$$\begin{aligned}
 & 10^{2^{(k+1)}-1} + 1 \\
 &= 10^{2^k+2-1} + 1 \\
 &= 10^{2^k+1} + 1 \\
 &= 10^2 (10^{2^k-1} + 1) + 1 \\
 &= 10^2 (10^{2^k-1} + 1) - 10^2 + 1 \\
 &= 10^2 \cdot 11m - 100 + 1 \quad \quad \quad [\text{Using (1)}] \\
 &= 100 \times 11m - 99 \\
 &= 11(100m - 9) \\
 &= 11r, \text{ where } r = (100m - 9) \text{ is some natural number}
 \end{aligned}$$

Therefore, $10^{2^{(k+1)}-1} + 1$ is divisible by 11.

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 21:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$: $x^{2^n} - y^{2^n}$ is divisible by $x + y$.

Answer

Let the given statement be $P(n)$, i.e.,

$P(n)$: $x^{2^n} - y^{2^n}$ is divisible by $x + y$.

It can be observed that $P(n)$ is true for $n = 1$.

This is so because $x^{2 \times 1} - y^{2 \times 1} = x^2 - y^2 = (x + y)(x - y)$ is divisible by $(x + y)$.

Let $P(k)$ be true for some positive integer k , i.e.,

$x^{2k} - y^{2k}$ is divisible by $x + y$.

$$\therefore x^{2k} - y^{2k} = m(x + y), \text{ where } m \in \mathbf{N} \dots (1)$$

We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true.

Consider

$$\begin{aligned} & x^{2(k+1)} - y^{2(k+1)} \\ &= x^{2k} \cdot x^2 - y^{2k} \cdot y^2 \\ &= x^2(x^{2k} - y^{2k} + y^{2k}) - y^{2k} \cdot y^2 \\ &= x^2\{m(x + y) + y^{2k}\} - y^{2k} \cdot y^2 \quad [\text{Using (1)}] \\ &= m(x + y)x^2 + y^{2k} \cdot x^2 - y^{2k} \cdot y^2 \\ &= m(x + y)x^2 + y^{2k}(x^2 - y^2) \\ &= m(x + y)x^2 + y^{2k}(x + y)(x - y) \\ &= (x + y)\{mx^2 + y^{2k}(x - y)\}, \text{ which is a factor of } (x + y). \end{aligned}$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 22:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$: $3^{2n+2} - 8n - 9$ is divisible by 8.

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): 3^{2n+2} - 8n - 9 \text{ is divisible by } 8.$$

It can be observed that $P(n)$ is true for $n = 1$ since $3^{2 \times 1 + 2} - 8 \times 1 - 9 = 64$, which is divisible by 8.

Let $P(k)$ be true for some positive integer k , i.e.,

$$3^{2k+2} - 8k - 9 \text{ is divisible by } 8.$$

$$\therefore 3^{2k+2} - 8k - 9 = 8m; \text{ where } m \in \mathbf{N} \dots (1)$$

We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true.

Consider

$$\begin{aligned}
& 3^{2(k+1)+2} - 8(k+1) - 9 \\
&= 3^{2k+2} \cdot 3^2 - 8k - 8 - 9 \\
&= 3^2 (3^{2k+2} - 8k - 9 + 8k + 9) - 8k - 17 \\
&= 3^2 (3^{2k+2} - 8k - 9) + 3^2 (8k + 9) - 8k - 17 \\
&= 9 \cdot 8m + 9(8k + 9) - 8k - 17 \\
&= 9 \cdot 8m + 72k + 81 - 8k - 17 \\
&= 9 \cdot 8m + 64k + 64 \\
&= 8(9m + 8k + 8) \\
&= 8r, \text{ where } r = (9m + 8k + 8) \text{ is a natural number}
\end{aligned}$$

Therefore, $3^{2(k+1)+2} - 8(k+1) - 9$ is divisible by 8.

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 23:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$: $41^n - 14^n$ is a multiple of 27.

Answer

Let the given statement be $P(n)$, i.e.,

$P(n)$: $41^n - 14^n$ is a multiple of 27.

It can be observed that $P(n)$ is true for $n = 1$ since $41^1 - 14^1 = 27$, which is a multiple of 27.

Let $P(k)$ be true for some positive integer k , i.e.,

$41^k - 14^k$ is a multiple of 27

$\therefore 41^k - 14^k = 27m$, where $m \in \mathbf{N}$... (1)

We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true.

Consider

$$\begin{aligned}
& 41^{k+1} - 14^{k+1} \\
&= 41^k \cdot 41 - 14^k \cdot 14 \\
&= 41(41^k - 14^k + 14^k) - 14^k \cdot 14 \\
&= 41(41^k - 14^k) + 41 \cdot 14^k - 14^k \cdot 14 \\
&= 41 \cdot 27m + 14^k(41 - 14) \\
&= 41 \cdot 27m + 27 \cdot 14^k \\
&= 27(41m + 14^k) \\
&= 27 \times r, \text{ where } r = (41m + 14^k) \text{ is a natural number}
\end{aligned}$$

Therefore, $41^{k+1} - 14^{k+1}$ is a multiple of 27.

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .

Question 24:

Prove the following by using the principle of mathematical induction for all $n \in \mathbf{N}$:

$$(2n + 7) < (n + 3)^2$$

Answer

Let the given statement be $P(n)$, i.e.,

$$P(n): (2n + 7) < (n + 3)^2$$

It can be observed that $P(n)$ is true for $n = 1$ since $2 \cdot 1 + 7 = 9 < (1 + 3)^2 = 16$, which is true.

Let $P(k)$ be true for some positive integer k , i.e.,

$$(2k + 7) < (k + 3)^2 \dots (1)$$

We shall now prove that $P(k + 1)$ is true whenever $P(k)$ is true.

Consider

$$\{2(k+1)+7\} = (2k+7)+2$$

$$\therefore \{2(k+1)+7\} = (2k+7)+2 < (k+3)^2 + 2 \quad [\text{using (1)}]$$

$$2(k+1)+7 < k^2 + 6k + 9 + 2$$

$$2(k+1)+7 < k^2 + 6k + 11$$

$$\text{Now, } k^2 + 6k + 11 < k^2 + 8k + 16$$

$$\therefore 2(k+1)+7 < (k+4)^2$$

$$2(k+1)+7 < \{(k+1)+3\}^2$$

Thus, $P(k + 1)$ is true whenever $P(k)$ is true.

Hence, by the principle of mathematical induction, statement $P(n)$ is true for all natural numbers i.e., n .