Complete Study Guide & Notes On
ALGEBRA OF MATRICES & DETERMINANTS

A Formulae Guide By OP Gupta (Indira Award Winner)

If I have the belief that I can do it, I will acquire all the capacity to do it even if I may not have it at the beginning!

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Matrix - a basic introduction: A matrix is an ordered rectangular array of numbers (real or complex) or functions which are known as *elements* or the *entries* of the matrix. It is denoted by the **upper case letters** *i.e.* A, B, C etc.

Consider a matrix A given as, $A=\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1j} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2j} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \ldots & a_{ij} & \ldots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mj} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$.

Here in matrix A depicted above, the horizontal lines of elements are said to constitute ***rows*** of the matrix A and vertical lines of elements are said to constitute ***columns*** of the matrix. Thus matrix A has ***m* rows** and ***n* columns**. The array is enclosed by brackets $\begin{bmatrix} & \end{bmatrix}$, the parentheses $(\ )$ and the double vertical bars $\|\ \|$.

- *A matrix having m rows and n columns is called a matrix of order $m\times n$ (read as 'm by n' matrix). And a matrix A of order $m\times n$ is depicted as $A=\left[a_{ij}\right]_{m\times n}$; $i, j\in \mathbb{N}$.*
- *Also in general, a_{ij} means an element lying in the i^{th} row and j^{th} column.*
- *No. of elements in the matrix $A=\left[a_{ij}\right]_{m\times n}$ is given as $(m)(n)$.*

02. Types of Matrices:

a) Column matrix: A matrix having only one column is called a *column matrix* or *column vector*.

e.g. $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}_{3\times 1}$, $\begin{bmatrix} 8 \\ 5 \end{bmatrix}_{2\times 1}$.

➲ **General notation:** $A=\left[a_{ij}\right]_{m\times 1}$.

b) Row matrix: A matrix having only one row is called a *row matrix* or *row vector*.

e.g. $\begin{bmatrix} -1 & 2 & \sqrt{3} & 4 \end{bmatrix}_{1\times 4}$, $\begin{bmatrix} 2 & 5 & 0 \end{bmatrix}_{1\times 3}$

➲ **General notation:** $A=\left[a_{ij}\right]_{1\times n}$.

c) Square matrix: It is a matrix in which the number of rows is equal to the number of columns *i.e.*, an $m\times n$ matrix is said to constitute a square matrix if $m=n$ and is known as a ***square matrix of order 'n'***.

e.g. $\begin{bmatrix} 1 & 2 & 5 \\ 3 & 7 & -4 \\ 0 & -1 & -2 \end{bmatrix}_{3\times 3}$ is a square matrix of order 3.

➲ **General notation:** $A=\left[a_{ij}\right]_{n\times n}$.

d) Diagonal matrix: A square matrix $A=\left[a_{ij}\right]_{m\times m}$ is said to be a *diagonal matrix* if $a_{ij}=0$, when $i\neq j$ *i.e.*, all its non- diagonal elements are *zero*.

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 4 \end{bmatrix}_{3\times3}$ is a diagonal matrix of order 3.

➢ *Also there is* ***one more notation*** *specifically used for the diagonal matrices. For instance, consider the matrix depicted above, it can be also written as diag* $(2 \quad 5 \quad 4)$.

➢ *Note that the elements* $a_{11}, a_{22}, a_{33}, ..., a_{mm}$ *of a square matrix* $A = \left[a_{ij}\right]_{m\times n}$ *of* ***order m*** *are said to constitute the* **principal diagonal** *or simply* ***the diagonal of the square matrix A****. And these elements are known as* ***diagonal elements of matrix A****.*

e) Scalar matrix: A *diagonal matrix* $A = \left[a_{ij}\right]_{m\times m}$ is said to be a *scalar matrix* if its *diagonal elements* are **equal** *i.e.*, $a_{ij} = \begin{cases} 0, & \text{when } i \neq j \\ k, & \text{when } i = j \text{ for some constant } k \end{cases}$.

e.g. $\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}_{3\times3}$ is a scalar matrix of order 3.

f) Unit or Identity matrix: A square matrix $A = \left[a_{ij}\right]_{m\times n}$ is said to be an *identity matrix* if $a_{ij} = \begin{cases} 1, \text{if } i = j \\ 0, \text{if } i \neq j \end{cases}$.

A *unit matrix* can also be defined as the *scalar matrix* each of whose diagonal elements is *unity*. We denote the identity matrix of order m by I_m or I.

e.g. $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

g) Zero matrix or Null matrix: A matrix is said to be a *zero matrix* or *null matrix* if each of its elements is '0'.

e.g. $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 \end{bmatrix}$.

h) Horizontal matrix: A $m\times n$ matrix is said to be a *horizontal matrix* if *m* < *n*.

e.g. $\begin{bmatrix} 1 & 2 & 0 \\ 5 & 4 & 7 \end{bmatrix}_{2\times3}$.

i) Vertical matrix: A $m\times n$ matrix is said to be a *vector matrix* if *m* > *n*.

e.g. $\begin{bmatrix} 2 & 5 \\ 0 & 7 \\ 3 & 1 \end{bmatrix}_{3\times2}$.

j) Triangular matrix:

Lower triangular matrix: A square matrix is called a lower triangular matrix if $a_{ij} = 0$ *when* $i < j$.

e.g. $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$, $\begin{bmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 5 & 7 \end{bmatrix}$.

Upper triangular matrix: A square matrix is called an upper triangular matrix if $a_{ij} = 0$ *when* $i > j$.

e.g. $\begin{bmatrix} 1 & 2 & 4 \\ 0 & 5 & 8 \\ 0 & 0 & 3 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 9 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

03. Equality of Matrices: Two matrices A and B are said to be equal and written as A = B, if they are of the **same orders** and their **corresponding elements are identical** *i.e.* $a_{ij} = b_{ij}$ for all *i* and *j*. That is $a_{11} = b_{11}$, $a_{22} = b_{22}$, $a_{32} = b_{32}$ etc.

04. Addition of matrix: If A and B are two $m\times n$ matrices, then another $m\times n$ matrix obtained by adding the corresponding elements of the matrices A and B is called the sum of the matrices A and B and is denoted by 'A + B'.

Thus if $A = \left[a_{ij}\right]$, $B = \left[b_{ij}\right] \Rightarrow A + B = \left[a_{ij} + b_{ij}\right]$.

➲ **Properties of matrix addition:**

- *Commutative property:* $A+B=B+A$
- *Associative property:* $A+(B+C)=(A+B)+C$
- *Cancellation laws:* ***i)*** *Left cancellation* - $A+B=A+C \Rightarrow B=C$
  ii) *Right cancellation* - $B+A=C+A \Rightarrow B=C$.

05. Multiplication of a matrix by a scalar: If an $m\times n$ matrix A is multiplied by a scalar k (say), then the new kA matrix is obtained by multiplying each element of matrix A by scalar k. Thus if $A=\left[a_{ij}\right]$ and it is multiplied by a scalar k then, $kA=\left[k\,a_{ij}\right]$, *i.e.,* $A=\left[a_{ij}\right]\Rightarrow kA=\left[k\,a_{ij}\right]$.

e.g. $A=\begin{bmatrix}2 & -1\\ 6 & 4\end{bmatrix}\Rightarrow 3A=\begin{bmatrix}6 & -3\\ 18 & 12\end{bmatrix}$.

06. Multiplication of two matrices: Let $A=\left[a_{ij}\right]$ be a $m\times n$ matrix and $B=\left[b_{jk}\right]$ be a $n\times p$ matrix such that the number of columns in A is equal to the number of rows in B, then the $m\times p$ matrix $C=\left[c_{ik}\right]$ such that $C_{ik}=\sum_{j=1}^{n} a_{ij}b_{jk}$ is said to be the product of the matrices A and B in that order and it is denoted by AB *i.e.* "C = AB'.

[It might be that you wouldn't have understood this matrix multiplication as explained here! But don't worry, it's extremely simple and, can be easily understood once you go through the examples section.]

➲ **Properties of matrix multiplication:**

- Note that the **product AB is defined *only* when** the number of columns in matrix A is equal to the number of rows in matrix B.
- If A and B are $m\times n$ and $n\times p$ matrices respectively then the matrix AB will be an $m\times p$ matrix *i.e.,* order of matrix AB will be $m\times p$.
- In the product AB, A is called the ***pre-factor*** and B is called the ***post-factor***.
- If two matrices A and B are such that AB is possible then it is ***not necessary*** that the product BA is also possible.
- If A is a $m\times n$ matrix and both AB as well as BA are defined then B will be a $n\times m$ matrix.
- If A is a $n\times n$ matrix and I_n be the unit matrix of order n then, $A\,I_n=I_n\,A=A$.
- Matrix multiplication is ***associative*** *i.e.*, $A(BC)=(AB)C$.
- Matrix multiplication is ***distributive over the addition*** *i.e.*, $A(B+C)=AB+AC$.

➲ **Idempotent matrix:** A square matrix A is said to be an idempotent matrix if $\mathbf{A^2=A}$.

For example, $\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$.

07. Transpose of a Matrix: If $A=\left[a_{ij}\right]_{m\times n}$ be an $m\times n$ matrix, then the matrix obtained by interchanging the rows and columns of matrix A is said to be a ***transpose of matrix A***. The transpose of A is denoted by A' or A^T or A^c *i.e.,* if $A=\left[a_{ij}\right]_{m\times n}$ then, $A^T=\left[a_{ji}\right]_{n\times m}$.

For example, $\begin{bmatrix}3 & 2 & 0\\ 1 & -2 & 6\end{bmatrix}^T=\begin{bmatrix}3 & 1\\ 2 & -2\\ 0 & 6\end{bmatrix}$.

➲ **Properties of Transpose of matrices:**

- $(A+B)^T=A^T+B^T$
- $(A-B)^T=A^T-B^T$

- $(A^T)^T = A$
- $(kA)^T = kA^T$ where, k is any constant
- $(AB)^T = B^TA^T$
- $(ABC)^T = C^TB^TA^T$

08. Symmetric matrix: A square matrix $A = \left[a_{ij}\right]$ is said to be a *symmetric matrix* if $A^T = A$.

That is, if $A = \left[a_{ij}\right]$ then, $A^T = \left[a_{ji}\right] = \left[a_{ij}\right] \Rightarrow A^T = A$.

For example: $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}, \begin{bmatrix} 2+i & 1 & 3 \\ 1 & 2 & 3+2i \\ 3 & 3+2i & 4 \end{bmatrix}$.

09. Skew-symmetric matrix: A square matrix $A = \left[a_{ij}\right]$ is said to be a *skew-symmetric matrix* if $A^T = -A$ *i.e.*, if $A = \left[a_{ij}\right]$ then, $A^T = \left[a_{ji}\right] = -\left[a_{ij}\right] \Rightarrow A^T = -A$.

For example: $\begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$.

Facts you should know:

- Note that $\left[a_{ji}\right] = -\left[a_{ij}\right] \Rightarrow \left[a_{ii}\right] = -\left[a_{ii}\right] \Rightarrow 2\left[a_{ii}\right] = 0$ (Replacing j by i)
  That is, ***all the diagonal elements in a skew-symmetric matrix are zero***.
- The matrices AA^T and A^TA are symmetric matrices.
- For any square matrix A, the matrix A + A^T is a symmetric matrix and A – A^T is a skew-symmetric matrix *always*.
- Also note that ***any square matrix can be expressed as the sum of a symmetric and a skew-symmetric matrix*** *i.e.*, $A = \frac{1}{2}(P) + \frac{1}{2}(Q)$, where P = A + A^T is a symmetric matrix and Q = A – A^T is a skew-symmetric matrix.

10. Orthogonal matrix: A matrix A is said to be orthogonal if $A.A^T = I$ where A^T is transpose of A.

11. Invertible Matrix: If A is a square matrix of order *m* and if there exists another square matrix B of the same order *m*, such that AB = BA = I, then B is called the inverse matrix of A and it is denoted by A^{-1}. A matrix having an inverse is said to be ***invertible***.

➢ *It is to note that if B is inverse of A, then A is also the inverse of B. In other words, if it is known that* $AB = BA = I$ *then,* $A^{-1} = B \Leftrightarrow B^{-1} = A$.

12. Determinants, Minors & Cofactors:

a) Determinant: A unique number (real or complex) can be associated to every square matrix $A = \left[a_{ij}\right]$ of order *m*. This number is called the determinant of the square matrix A, where $a_{ij} = (i, j)^{th}$ element of A.

For instance, if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then, determinant of matrix A is written as $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det.(A)$ and its value is given by $ad - bc$.

b) Minors: Minors of an element a_{ij} of a determinant *(or a determinant corresponding to matrix A)* is the determinant obtained by deleting its i^{th} row and j^{th} column in which a_{ij} lies. Minor of a_{ij} is denoted by M_{ij}. Hence we can get 9 minors corresponding to the 9 elements of a third order (*i.e.*, 3×3) determinant.

c) Cofactors: Cofactor of an element a_{ij}, denoted by A_{ij}, is defined by, $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is minor of a_{ij}. Sometimes C_{ij} is used in place of A_{ij} to denote the cofactor of element a_{ij}.

13. Adjoint of a square matrix: Let $A=\left[a_{ij}\right]$ be a square matrix. Also assume $B=\left[A_{ij}\right]$ where A_{ij} is the cofactor of the elements a_{ij} in matrix A. Then the transpose B^T of matrix B is called the ***adjoint of matrix A*** and it is denoted by "***adj.*A**".

➢ ***To find adjoint of a*** 2×2 ***matrix:*** Follow this, $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow adj.A=\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

For example, consider a square matrix of order 3 as $A=\begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 2 & 0 & 5 \end{bmatrix}$ then, in order to find the adjoint of matrix A, we find a matrix B (formed by the cofactors of elements of matrix A as mentioned above in the definition)

i.e., $B=\begin{bmatrix} 15 & -2 & -6 \\ -10 & -1 & 4 \\ -1 & 2 & -1 \end{bmatrix}$. Hence, $adj.A=B^T=\begin{bmatrix} 15 & -10 & -1 \\ -2 & -1 & 2 \\ -6 & 4 & -1 \end{bmatrix}$.

14. Singular matrix & Non-singular matrix:

a) Singular matrix: A square matrix A is said to be *singular if* $|A|=0$ *i.e.*, its **determinant is *zero***.

e.g. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 12 \\ 1 & 1 & 3 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ 3 & -4 \end{bmatrix}$.

b) Non-singular matrix: A square matrix A is said to be *non-singular if* $|A|\neq 0$.

e.g. $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}$.

❖ *A square matrix A is **invertible** if and only if A is **non-singular**.*

15. Elementary Operations or Transformations of a Matrix: The following three operations applied on the row (or column) of a matrix are called elementary row (or column) transformations-

a) *Interchange of any two rows (or columns)****:*** When i^{th} row (or column) of a matrix is interchanged with the j^{th} row (or column), it is denoted as $R_i \leftrightarrow R_j$ (or $C_i \leftrightarrow C_j$).

b) *Multiplying all elements of a row (or column) of a matrix by a non-zero scalar****:*** When the i^{th} row (or column) of a matrix is multiplied by a scalar *k*, it is denoted as $R_i \to kR_i$ (or $C_i \to kC_i$).

c) *Adding to the elements of a row (or column), the corresponding elements of any other row (or column) multiplied by any scalar k****:*** When *k* times the elements of j^{th} row (or column) is added to the corresponding elements of the i^{th} row (or column), it is denoted as $R_i \to R_i + kR_j$ (or $C_i \to C_i + kC_j$).

> **NOTE:** In case, after applying one or more elementary row (or column) operations on A = IA (or A = AI), if we obtain **all zeros in one or more rows** of the matrix A on LHS, then $\mathbf{A^{-1}}$ **does not exist.**

16. Inverse or reciprocal of a square matrix: If A is a square matrix of order n, then a matrix B (if such a matrix exists) is called the inverse of A if $AB=BA=I_n$. Also note that the inverse of a square matrix A is denoted by A^{-1} and we write, $A^{-1}=B$.

➢ *Inverse of a square matrix A exists if and only if A is non-singular matrix i.e.,* $|A|\neq 0$.

➢ *If B is inverse of A, then A is also the inverse of B.*

17. Algorithm to find Inverse of a matrix by Elementary Operations or Transformations:

➲ **By Row Transformations:**

STEP1- Write the given square matrix as $A = I_nA$.

STEP2- Perform a sequence of elementary row operations successively on A on the LHS and pre-factor I_n on the RHS till we obtain the result $I_n = BA$.

STEP3- Matrix B is the inverse of A. So, write $A^{-1} = B$.

➲ **By Column Transformations:**

STEP1- Write the given square matrix as $A = AI_n$.

STEP2- Perform a sequence of elementary column operations successively on A on the LHS and post-factor I_n on the RHS till we obtain the result $I_n = AB$.

STEP3- Matrix B is the inverse of A. So, write $A^{-1} = B$.

18. Algorithm to find A^{-1} by Determinant method:

STEP1- Find |A|.

STEP2- If |A| = 0 then, write **"A is a singular matrix and hence not invertible"**. Else write **"A is a non-singular matrix and hence invertible"**.

STEP3- Calculate the cofactors of elements of matrix A.

STEP4- Write the matrix of cofactors of elements of A and then obtain its transpose to get *adj.*A (*i.e.*, *adjoint* A).

STEP5- Find the inverse of A by using the relation $A^{-1} = \frac{1}{|A|} adj.A$.

19. Properties associated with various operations of Matrices & the Determinants:

a) $AB = I = BA$

b) $AA^{-1} = I$ or $A^{-1}I = A^{-1}$

c) $(AB)^{-1} = B^{-1}A^{-1}$

d) $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

e) $\left(A^{-1}\right)^{-1} = A$

f) $\left(A^{T}\right)^{-1} = \left(A^{-1}\right)^{T}$

g) $A(adj.A) = (adj.A)A = |A|\,I$

h) $adj.(AB) = (adj.B)(adj.A)$

i) $adj.\left(A^{T}\right) = (adj.A)^{T}$

j) $(adj.A)^{-1} = \left(adj.A^{-1}\right)$

k) $|adj.A| = |A|^{n-1}$, *if* $|A| \neq 0$, where *n* is order of A

l) $|AB| = |A||B|$

m) $|A.\ adj.A| = |A|^n$, where *n* is order of A

n) $\left|A^{-1}\right| = \frac{1}{|A|}$, iff matrix A is invertible

o) $|A| = |A^T|$

- $|kA| = k^n |A|$ where *n* is order of square matrix *A* and *k* is any scalar.
- If A is a non-singular matrix (*i.e.*, when $|A| \neq 0$) of order *n*, then $|adj.A| = |A|^{n-1}$.
- If A is a non-singular matrix of order *n*, then $adj.(adj.A) = |A|^{n-2} A$.

20. Properties of Determinants:

a) If any two rows or columns of a determinant are *proportional* or *identical*, then its value is equal to *zero*.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$ [As R_1 and R_3 are the same.

b) The value of a determinant remains *unchanged* if its rows and columns are *interchanged*.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Here rows and columns have been interchanged, but there is *no effect* on the value of determinant.

c) If each element of a row or a column of a determinant is multiplied by a constant *k*, then the value of new determinant is *k* times the value of the original determinant.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} ka_1 & kb_1 & kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = k\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow \Delta_1 = k\Delta$.

d) If *any two* rows or columns are *interchanged*, then the determinant retains its *absolute* value, but its *sign* is changed.

e.g. $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, $\Delta_1 = \begin{vmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} \Rightarrow \Delta_1 = -\Delta$ [Here $R_1 \leftrightarrow R_3$.

e) If every element of some column or row is the *sum* of two terms, then the determinant is equal to the sum of two determinants; one containing only the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as given in the original determinant.

e.g. $\Delta = \begin{vmatrix} a_1+\alpha & b_1 & c_1 \\ a_2+\beta & b_2 & c_2 \\ a_3+\gamma & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha & b_1 & c_1 \\ \beta & b_2 & c_2 \\ \gamma & b_3 & c_3 \end{vmatrix}$.

21. Area of triangle: Area of a triangle whose vertices are (x_1, y_1), (x_2, y_2) and (x_3, y_3) is given by,

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \text{ Sq. units}. \quad \ldots(A)$$

➲ *Since area is a positive quantity, we take **absolute value of the determinant** in (A).*

➲ *If the points (x_1, y_1), (x_2, y_2) and (x_3, y_3) are **collinear** then $\Delta = 0$.*

➲ *The **equation of a line** passing through the points (x_1, y_1) and (x_2, y_2) can be obtained by the expression given here:*

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

22. Solutions of System of Linear equations:

a) Consistent and Inconsistent system: A system of equations is consistent if it has *one or more* solutions otherwise it is said to be an inconsistent system. In other words an inconsistent system of equations has *no solution*.

b) Homogeneous and Non-homogeneous system: A system of equations $AX = B$ is said to be a homogeneous system if $B = 0$. Otherwise it is called a non-homogeneous system of equations.

23. Solving of system of equations by Matrix method [Inverse Matrix Method]:

Consider the following system of equations,

$$a_1x + b_1y + c_1z = d_1,$$
$$a_2x + b_2y + c_2z = d_2,$$
$$a_3x + b_3y + c_3z = d_3.$$

STEP1- Assume $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$ and $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

STEP2- Find |A|. Now there may be following situations:

a) $|A| \neq 0 \Rightarrow A^{-1}$ exists. It implies that the given system of equations is *consistent* and therefore, the system has ***unique solution***. In that case, write

$$AX = B$$

$\Rightarrow \quad X = A^{-1}B \qquad \left[\text{where } A^{-1} = \frac{1}{|A|}(adj.A)\right.$

$\Rightarrow$ Then by using the ***definition of equality of matrices***, we can get the values of *x, y* and *z*.

b) $|A| = 0 \Rightarrow A^{-1}$ does not exist. It implies that the given system of equations may be *consistent* or *inconsistent*. In order to check proceed as follow:

$\Rightarrow$ Find (*adj*.A)B. Now we may have either $(adj.A)B \neq 0$ or $(adj.A)B = 0$.

- If $(adj.A)B = 0$, then the given system may be *consistent* or *inconsistent*.

  To check, put $z = k$ in the given equations and proceed in the same manner in the new *two variables* system of equations assuming $d_i - c_i k$, $1 \leq i \leq 3$ as constant.

- And if $(adj.A)B \neq 0$, then the given system is *inconsistent* with *no solutions*.

Any queries and/or suggestion(s), please write to me at

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Please mention your details : Name, Student/Teacher/Tutor, School/Institution, Place, Contact No. (if you wish)