Complete Study Guide & Notes On
GENERAL USEFUL FORMULAE

A Formulae Guide By OP Gupta (Indira Award Winner)

If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Sum of Infinite Geometric Progression:

If S_∞ denotes the sum to infinity of a GP then, $S_\infty = \dfrac{a}{1-r}$ when $-1 < r < 1$,

where a be the first term of Infinite Geometric Progression and r be its common ratio.

02. Factorial notation: The product of first n natural numbers is denoted by $n!$ or $\underline{|n}$ and is read as 'factorial n'.

Thus, $n! = 1\times 2\times 3\times 4\times \ldots \times (n-1)\times n = n\times (n-1)\times (n-2)\times \ldots \times 3\times 2\times 1$.

➲ *Note that we define $0! = 1$ and also the factorials of proper fractions or negative integers are not defined.*

03. The number of permutations of n different things taken r at a time: It is given by $^n P_r = P(n,r) = \dfrac{n!}{(n-r)!}$ such that $0 \le r \le n$. Counting permutations is just counting the number of ways in which some or all the objects at a time are rearranged. Arranging no object at all is the same as leaving behind all the objects and we can easily deduce that there is only one way of doing so. Thus we have, $^n P_0 = \dfrac{n!}{(n-0)!} = \dfrac{n!}{n!} = 1$.

Also, the number of permutations (*i.e.*, arrangements) of n different things taken r at a time, where repetition is allowed, is $n\times n\times \ldots\ r$ times $= n^r$.

04. The number of combinations of n different things taken r at a time: It is given by $^n C_r$, which is defined as $^n C_r = C(n,r) = \dfrac{n!}{r!(n-r)!}$, $0 \le r \le n$.

➲ **Note the followings:**

- $^n P_r = {}^n C_r \times r!$, $0 < r \le n$.
- $^n C_n = {}^n C_0 = 1$.
- $^n C_{n-r} = \dfrac{n!}{(n-r)!\big(n-(n-r)\big)!} = \dfrac{n!}{(n-r)!\,r!} = {}^n C_r \Rightarrow {}^n C_r = {}^n C_{n-r}$.
- $^n C_x = {}^n C_y \Rightarrow x = y$ or $x = n - y$ *i.e.*, $n = x + y$.
- $^n C_r + {}^n C_{r-1} = {}^{n+1} C_r$.

05. Indices (Laws Of Exponents):

a) $a^m . a^n = a^{m+n}$ **b)** $\dfrac{a^m}{a^n} = a^{m-n}$ **c)** $\left(a^m\right)^n = a^{mn}$ **d)** $a^{-n} = \dfrac{1}{a^n}$

e) $a^0 = 1$ **f)** $a^m . b^m = (ab)^m$ **g)** $\left(\dfrac{a}{b}\right)^m = \left(\dfrac{b}{a}\right)^{-m}$.

06. Componendo & Dividendo:

If $\dfrac{a}{b} = \dfrac{c}{d}$ then we have, $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$ and $\dfrac{a-b}{a+b} = \dfrac{c-d}{c+d}$.

NUMBER SYSTEM

01. Natural numbers: The numbers used in ordinary counting i.e. $1, 2, 3, ...,$ are called natural numbers (and *positive integers* as well). The set of natural nos. is denoted by N. Also if we include 0 to the set of natural numbers, we get set of the *whole numbers* which is denoted by the symbol W.

Therefore $\mathrm{N}=\{1,2,3,...\}$ and, $\mathrm{W}=\{0,1,2,3,...\}$.

02. Integers: The numbers $...-3,-2,-1,0,1,2,3,...$ are called integers. The set of integers is denoted by I *or* Z. Though now we use Z instead of I to symbolize the set of integers.

Therefore, I *or* $\mathrm{Z}=\{...-3,-2,-1,0,1,2,3,...\}$.

Clearly $\mathrm{N}\subset\mathrm{Z}$.

- Also from the above discussion, it is evident that integers are of three types viz.:

a) *Positive integers i.e.* $\mathrm{Z}^{+}=\{1,2,3,...\}$

b) *Negative integers i.e.* $\mathrm{Z}^{-}=\{-1,-2,-3,...\}$

c) *Zero integer i.e. non-positive and non-negative integer.*

03. Rational numbers: A number of the form $\frac{p}{q}$, where p and q are integers and $q\neq 0$, is called a rational no. The set of rational nos. is denoted by Q.

Therefore $\mathrm{Q}=\left\{\frac{p}{q};\, p,q\in Z \text{ and } q\neq 0\right\}$

Clearly $\mathrm{N}\subset\mathrm{Z}\subset\mathrm{Q}$.

- Zero being an integer, is also a rational number.

04. Irrational numbers: An irrational number has a non-terminating and non-repeating decimal representation *i.e.* it can't be expressed in the form of $\frac{p}{q}$. The set of irrational nos. is denoted by T.

Few examples of irrational numbers are $\sqrt{2},\ 5\sqrt{7},\ 8+\sqrt{3}, \sqrt[3]{5}, e,\ \pi, ...$ *etc.*

- Note that π is irrational while $\frac{22}{7}$ is rational.

05. Real numbers: The set of all numbers either rational or irrational, is called real number. The set of real nos. is denoted by R.

Clearly $\mathrm{N}\subset\mathrm{Z}\subset\mathrm{Q}\subset\mathrm{R}$.

➲ Solving of a Quadratic Equation:

Consider a quadratic equation of the form, $ax^2+bx+c=0$ then, its roots are given by

$$x=\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and, } x=\frac{-b-\sqrt{b^2-4ac}}{2a} \quad \text{where } D=b^2-4ac.$$

➲ Logarithmic Relations:

a) $log_a p=\dfrac{log_b p}{log_b a}$

b) $log_a p=\dfrac{1}{log_p a}$

c) $log_a\left(m^n\right)=nlog_a\left(m\right)$

d) $log_a\left(m.n\right)=log_a\left(m\right)+log_a\left(n\right)$

e) $log_a\left(\dfrac{m}{n}\right)=log_a\left(m\right)-log_a\left(n\right)$

f) $log_a a=1$

g) $log_a a^p=plog_a a=p$

h) $a^{log_a f(x)}=f(x)$.

➲ Exponential Series:

a) $a^x = 1 + x.(log_e a) + \frac{x^2}{2!}.(log_e a)^2 + ... + \frac{x^n}{n!}.(log_e a)^n + ...\infty$

b) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... + \frac{x^n}{n!} + ...\infty$ for all x.

➲ Logarithmic Series:

a) $log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...\infty, \ -1 < x \le 1$

b) $log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - ...\infty, \ -1 \le x < 1$.

➲ Binomial Expansions:

a) $(a+b)^n = {}^nC_0 a^{n-0} b^0 + {}^nC_1 a^{n-1} b^1 + {}^nC_2 a^{n-2} b^2 + ... + {}^nC_r a^{n-r} b^r + ... + {}^nC_n a^{n-n} b^n$ if $n \in Z^+$.

b) $(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + ... + {}^nC_r x^r + ... + {}^nC_n x^n$ if n is a *positive integer*.

c) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...\infty$ such that $-1 < x < 1$ and $n \in Z^-$ or $n \in Q$.

d) $\frac{x^n - a^n}{x - a} = x^{n-1} + a.x^{n-2} + a^2.x^{n-3} + ... + a^{n-1}$.

➲ Trigonometric Series:

a) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ...\infty$

b) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...\infty$

c) $\tan x = x + \frac{x^3}{3} + \frac{2}{15}x^5 + ...\infty$.

➲ Sum of Special Sequences:

a) Sum of first n natural numbers: $1 + 2 + 3 + ... + n = \sum_{r=1}^{n} r = \frac{n(n+1)}{2}$

b) Sum of squares of first n natural numbers: $1^2 + 2^2 + 3^2 + ... + n^2 = \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$

c) Sum of cubes of first n natural numbers: $1^3 + 2^3 + 3^3 + ... + n^3 = \sum_{r=1}^{n} r^3 = \left[\frac{n(n+1)}{2}\right]^2$

d) Sum of any constant k to n times: $k + k + k + ... + k(\text{to } n \text{ times}) = \sum_{r=1}^{n} k = nk$.

➲ List of formulae of Mensuration for using in the problems of Maxima and Minima:

01. Circle:
Perimeter *i.e.* circumference $= 2\pi r$
Area $= \pi r^2$

02. Equilateral triangle:
Perimeter $= 3a$
Area $= \frac{\sqrt{3}}{4}a^2$

03. Rectangle:
Perimeter $= 2(l+b)$
Area $= lb$

04. Square:
Perimeter $= 4a$
Area $= a^2$

05. Cuboid:
Lateral surface area $= 2(l+b).h$
Total surface area $= 2(lb+bh+hl)$
Volume $= lbh$

06. Cube:
Lateral surface area $= 4a^2$
Total surface area $= 6a^2$
Volume $= a^3$

07. Sphere:

Surface area $= 4\pi r^2$

Volume $= \dfrac{4}{3}\pi r^3$

08. Hemisphere:

Curved surface area $= 2\pi r^2$

Total surface area $= 3\pi r^2$

Volume $= \dfrac{2}{3}\pi r^3$

09. Cylinder:

Curved surface area $= 2\pi rh$

Total surface area $= 2\pi r^2 + 2\pi rh$

Volume $= \pi r^2 h$

10. Cone:

Curved surface area $= \pi rl$, *where* $l^2 = r^2 + h^2$

Total surface area $= \pi r^2 + \pi rl$

Volume $= \dfrac{1}{3}\pi r^2 h$

☞ Followings are also of **importance**, though questions on them are **rarely** found in maxima and minima:

11. Frustum of a cone:

Curved surface area $= \pi l\left(R + r\right)$, *where* $l = \sqrt{h^2 + \left(R - r\right)^2}$

Total surface area $= \pi l\left(R + r\right) + \pi\left(R^2 + r^2\right)$, *where* $l = \sqrt{h^2 + \left(R - r\right)^2}$

Volume $= \dfrac{1}{3}\pi h\left(R^2 + r^2 + rR\right)$

12. Sector and segment of Circle:

Area of the sector of angle $\theta = \dfrac{\theta}{360} \times \pi r^2$

Length of the arc of a sector of angle $\theta = \dfrac{\theta}{360} \times 2\pi r$

Area of the segment of a circle $=$ Area of the corresponding sector $-$ Area of corresponding triangle

$$= \frac{\theta}{360} \times \pi r^2 - r^2 \sin\frac{\theta}{2}\cos\frac{\theta}{2} = \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin\theta$$

Circumference i.e. arc length of semicircle of radius r $= \pi r$

Perimeter of semicircle of radius $r = \pi r + 2r$

13. Area of a trapezium $= \dfrac{1}{2}$ (Sum of *parallel* sides) $\times$ (Distance between *parallel* sides)

14. Area of a rhombus $= \dfrac{1}{2}$ (Product of *diagonals*)

15. Area of a triangle ABC $= \dfrac{1}{2} ab\sin C = \dfrac{1}{2} bc\sin A = \dfrac{1}{2} ca\sin B$

16. Area of a triangle by Heron's formula $= \sqrt{s(s-a)(s-b)(s-c)}$ *where,* $s = \dfrac{a+b+c}{2}$.

➲ Algebraic Identities:

a) $\left(a \pm b\right)^2 = a^2 \pm 2ab + b^2$

b) $a^2 - b^2 = \left(a+b\right)\left(a-b\right)$

c) $\left(a+b\right)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

d) $\left(a-b\right)^3 = a^3 - 3a^2b + 3ab^2 - b^3$

e) $a^3 - b^3 = \left(a-b\right)\left(a^2 + ab + b^2\right)$

f) $a^3 + b^3 = \left(a+b\right)\left(a^2 - ab + b^2\right)$

g) $\left(a+b+c\right)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca$

h) $a^3 + b^3 + c^3 - 3abc = \left(a+b+c\right)\left(a^2 + b^2 + c^2 - ab - bc - ca\right)$.

➲ Concept of Infinity:

We consider the existence of **two symbols** $-\infty$ and ∞ **outside** the set of real numbers R and would call them ***minus infinity*** and ***plus infinity*** respectively with the fact $-\infty < x < \infty$ for every $x \in \mathrm{R}$.
Thus $-\infty$ and ∞ are not real numbers but just the symbols (like we use *x, y* etc.).

When we write $x = \infty$, we mean that:

a) x is larger than any real number however large.

b) x is not a fixed number.

Also $c^{\infty} = \begin{cases} \infty, \text{ if } c > 1 \\ 0, \text{ if } 0 \le c < 1 \\ 1, \text{ if } c = 1 \end{cases}$.

➲ Symbols and their meanings:

S. No.	Symbol	Meaning
01.	N	Set of natural numbers
02.	I *or* Z	Set of integers
03.	Q	Set of rational numbers
04.	T	Set of irrational numbers
05.	R	Set of real numbers
06.	C	Set of complex numbers
07.	$\in$	is an element of (or belongs to)
08.	$\notin$	is not an element of (or does not belong to)
09.	S *or* ξ *or* U	Universal set
10.	: *or* /	Such that
11.	ϕ	Empty set *or* Null set
12.	$\subseteq$	is subset of
13.	$\supseteq$	is superset of
14.	$\subset$	is proper subset of
15.	$\supset$	is proper superset of
16.	$\cup$	Union
17.	$\cap$	Intersection
18.	$\forall$	For all
19.	$\Rightarrow$	Implies
20.	$\Leftrightarrow$	if and only if

Any queries and/or suggestion(s), please write to me at
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Please mention your details : Name, Student/Teacher/Tutor, School/Institution, Place, Contact No. (if you wish)