Complete Study Guide & Notes On
PRINCIPLE OF MATHEMATICAL INDUCTION

A Formulae Guide By OP Gupta (Indira Award Winner)

Be happy. It's Maths time now!

IMPORTANT TERMS, DEFINITIONS & RESULTS

Mathematical induction offers a standard technique to prove a proposition (it is a statement which is either true or false) about natural numbers. Method of mathematical induction is widely used in proving identities, theorems, divisibility of an expression by a number or by another expression, inequalities etc.

An algorithmic approach for doing the sums based on the principle of mathematical induction is given below:

Let P(n) be a statement for $n \in \mathrm{N}$ such that
(i) P(1) is true for $n = 1$.
(ii) P($k+1$) is true, whenever P(k) is true.
Then, P(n) is true for all natural numbers n.

WORKED OUT ILLUSTRATIVE EXAMPLES

Ex01. Prove by the principle of mathematical induction that for all $n \in \mathrm{N}$:

$$\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}.$$

Sol. Let $\mathrm{P}(n): \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$.

Now, $\mathrm{P}(1): \frac{1}{(2\times1-1)(2\times1+1)}=\frac{1}{2\times1+1}$

Clearly, $\mathrm{P}(1): \frac{1}{1.3}=\frac{1}{3}$. So, P(1) is true.

Let us assume that P(k) is true for all $k \in \mathrm{N}$. Then,

$$\mathrm{P}(k): \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1} \qquad ...(i)$$

Now, we have to show that P($k+1$) is also true whenever P(k) is true *i.e.*,

$$\mathrm{P}(k+1): \frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}.$$

Considering LHS, $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k+1)(2k+3)}$

$$=\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+...+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \qquad \text{[By using (i)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{(k+1)}{(2k+3)}=\text{RHS}$$

$\therefore$ P($k+1$) is true.

Hence by the principle of mathematical induction, P(*n*) is always true for all the natural numbers *n*.

Ex02. Prove by the principle of mathematical induction that $n < 2^n$ for all $n \in \mathrm{N}$.

Sol. Let $\mathrm{P}(n) : n < 2^n$.

Now, $\mathrm{P}(1) : 1 < 2^1$

Clearly, $1 < 2$. So, $\mathrm{P}(1)$ is true.

Let us assume that P(*k*) is true for all $k \in \mathrm{N}$. Then,

$\mathrm{P}(k) : k < 2^k$ ...(*i*)

Now, we have to show that $\mathrm{P}(k+1)$ is also true whenever P(*k*) is true *i.e.*,

$\mathrm{P}(k+1) : (k+1) < 2^{k+1}$.

Considering P(*k*), $k < 2^k$ [By using (i)

$\Rightarrow 2k < 2 \times 2^k$ [Multiplying both the sides by 2

$\Rightarrow 2k < 2^{k+1}$

$\Rightarrow k + k < 2^{k+1}$

$\Rightarrow k + 1 < 2^{k+1}$ $[\because 1 \le k \Rightarrow k + 1 \le k + k$

$\therefore$ $\mathrm{P}(k+1)$ is true.

Hence by the principle of mathematical induction, P(*n*) is always true for all the natural numbers *n*.

Ex03. Using the principle of mathematical induction, show that $11^{n+2} + 12^{2n+1}$, where *n* is a natural number, is divisible by 133.

Sol. Let $\mathrm{P}(n) : 11^{n+2} + 12^{2n+1}$ is divisible by 133.

Now, $\mathrm{P}(1) : 11^{1+2} + 12^{2\times1+1} = 11^3 + 12^3 = 3059 = 23 \times 133$

Clearly, 3059 is divisible by 133. So, $\mathrm{P}(1)$ is true.

Let us assume that P(*k*) is true for all $k \in \mathrm{N}$. Then,

$\mathrm{P}(k) : 11^{k+2} + 12^{2k+1}$ is divisible by 133.

That is, $11^{k+2} + 12^{2k+1} = 133m$, where *m* is an integer ...(*i*)

Now, we have to show that $\mathrm{P}(k+1)$ is also true whenever P(*k*) is true *i.e.*,

$\mathrm{P}(k+1) : 11^{k+3} + 12^{2k+3}$ is divisible by 133.

As $11^{k+3} + 12^{2k+3} = 11 \times \left(11^{k+2} + 12^{2k+1}\right) + 133\left(12^{2k+1}\right)$

$\Rightarrow 11^{k+3} + 12^{2k+3} = 11 \times (133m) + 133\left(12^{2k+1}\right)$ [By using (i)

$\Rightarrow \quad = 133\left(11m + 12^{2k+1}\right)$

$\Rightarrow 11^{k+3} + 12^{2k+3} = 133p$, where $p = 11m + 12^{2k+1}$, $p \in \mathrm{N}$

$$
\begin{array}{r|l}
 & \quad 11 \\
11^{k+2} + 12^{2k+1} & 11^{k+3} + 12^{2k+3} \\
 & 11^{k+3} + 11.12^{2k+1} \\
 & - \quad - \\
\hline
 & 133.12^{2k+1}
\end{array}
$$

So, $11^{k+3} + 12^{2k+3}$ is divisible by 133.

$\therefore$ $\mathrm{P}(k+1)$ is true.

Hence by the principle of mathematical induction, P(*n*) is always true for all the natural numbers *n*.

Ex04. Using induction, show that $2.7^n + 3.5^n - 5$ is divisible by 24 such that *n* is a natural number.

Sol. Let $\mathrm{P}(n) : 2.7^n + 3.5^n - 5$ is divisible by 24.

Now, $\mathrm{P}(1) : 2.7^1 + 3.5^1 - 5 = 14 + 15 - 5 = 24 = 24 \times 1$

Clearly, 24 is divisible by 24. So, $\mathrm{P}(1)$ is true.

Let us assume that P(*k*) is true for all $k \in \mathrm{N}$. Then,

$\mathrm{P}(k) : 2.7^k + 3.5^k - 5$ is divisible by 24.

So, $2.7^k + 3.5^k - 5 = 24m, m \in \mathrm{N}$

$\Rightarrow 2.7^k = 24m + 5 - 3.5^k$ …*(i)*

Now, we have to show that $\mathrm{P}(k+1)$ is also true whenever P(*k*) is true *i.e.*,

$\mathrm{P}(k+1) : 2.7^{k+1} + 3.5^{k+1} - 5$ is divisible by 24.

Consider $2.7^{k+1} + 3.5^{k+1} - 5 = 2.7^k \times 7 + 5 \times 3.5^k - 5 = 7(24m + 5 - 3.5^k) + 15.5^k - 5$ [By (i)

$\Rightarrow \quad = 7(24m) + 35 - 21.5^k + 15.5^k - 5 = 7(24m) + 30 - 6.5^k$

$\Rightarrow \quad = 7(24m) - 6(5^k - 5) = 7(24m) - 6(4p), p \in \mathrm{N}$ $\quad$ [$5^k - 5$ is a multiple of 4 $\forall\ k \in \mathrm{N}$

$\Rightarrow \quad = 24[7m - p] = 24\lambda$, where $7m - p = \lambda \in \mathrm{N}$

So, $2.7^{k+1} + 3.5^{k+1} - 5$ is divisible by 24.

$\therefore$ P$(k+1)$ is true.

Hence by the principle of mathematical induction, P(n) is always true for all the natural numbers n.

EXERCISE FOR PRACTICE

By using the Principle of Mathematical Induction, prove the followings for all the natural numbers $n \in \mathrm{N}$ (**from Q01 to Q18**):

Q01. $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$

Q02. $1^2 + 2^2 + 3^2 + ... + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

Q03. $1^3 + 2^3 + 3^3 + ... + n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

Q04. $1^3 + 3^3 + 5^3 + ... + (2n-1)^3 = n^2(2n^2 - 1)$

Q05. $a + ar + ar^2 + ... + ar^{n-1} = a\left(\dfrac{1-r^n}{1-r}\right)$

Q06. $3.2^2 + 3^2.2^3 + 3^3.2^4 + ... + 3^n.2^{n+1} = \dfrac{12}{5}(6^n - 1)$

Q07. $1 + 4 + 4^2 + ... + 4^{n-1} = \dfrac{2^{2n} - 1}{3}$

Q08. $\left(1+\dfrac{1}{1}\right)\left(1+\dfrac{1}{2}\right)\left(1+\dfrac{1}{3}\right)...\left(1+\dfrac{1}{n}\right) = n + 1$

Q09. $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + ... + \dfrac{1}{2^n} = 1 - \dfrac{1}{2^n}$

Q10. $1.3 + 2.3^2 + 3.3^3 + ... + n \cdot 3^n = \dfrac{(2n-1)3^{n+1} + 3}{4}$

Q11. $\dfrac{1}{1.4} + \dfrac{1}{4.7} + \dfrac{1}{7.10} + ... + \dfrac{1}{(3n-2)(3n+1)} = \dfrac{n}{3n+1}$

Q12. $\left(1+\dfrac{3}{1}\right)\left(1+\dfrac{5}{4}\right)\left(1+\dfrac{7}{9}\right)...\left(1+\dfrac{(2n+1)}{n^2}\right) = (n+1)^2$

Q13. $1 + \dfrac{1}{1+2} + \dfrac{1}{1+2+3} + ... + \dfrac{1}{(1+2+3+...+n)} = \dfrac{2n}{n+1}$

Q14. $1^2 + 3^2 + 5^2 + ... + (2n-1)^2 = \dfrac{n(2n-1)(2n+1)}{3}$

Q15. $1.3 + 3.5 + 5.7 + ... + (2n-1)(2n+1) = \dfrac{n(4n^2 + 6n - 1)}{3}$

Q16. $\dfrac{1}{1.2.3} + \dfrac{1}{2.3.4} + ... + \dfrac{1}{n(n+1)(n+2)} = \dfrac{n(n+3)}{4(n+1)(n+2)}$

Q17. $1.2.3 + 2.3.4 + 3.4.5 + ... + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.

Q18. For all $n \geq 1$, prove that $\dfrac{1}{1.2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + ... + \dfrac{1}{n(n+1)} = \dfrac{n}{n+1}$.

If $n \in \mathrm{N}$, then by using Principle of Mathematical Induction show that (**from Q19 to Q31**):

Q19. $n(n+1)(n+5)$ is divisible by 6.

Q20. $11^{n+2} + 12^{2n+1}$ is divisible by 133.

Q21. $3^{2n} - 1$ is divisible by 8.

Q22. $7^{2n} + 2^{3n-3}.3^{n-1}$ is divisible by 25.

Q23. $10^n + 3.4^{n+2} + 5$ is divisible by 9.

Q24. $3^{4n+2} + 5^{2n+1}$ is divisible by 14.

Q25. $3^{2n+2} - 8n - 9$ is divisible by 64.

Q26. $5.2^{3n-2} + 3^{3n-1}$ is a multiple of 19.

Q27. $2.7^n + 3.5^n - 5$ is divisible by 24.

Q28. $10^{2n-1} + 1$ is divisible by 11.

Q29. $x^{2n} - y^{2n}$ is divisible by $x + y$.

Q30. $41^n - 14^n$ is a multiple of 27.

Q31. $n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9.

Q32. Prove that $n^7 - n$ is a multiple of 7 for all $n \geq 2$.

Q33. Prove that $4^n - 3n - 1$ is divisible by 9 for all $n \geq 2$.

Q34. Prove that $n\left(n^2 + 20\right)$ is divisible by 48 where n is an even positive integer.

Q35. By using the principle of mathematical induction prove that $n^2 + n$ is even natural number for all $n \in \text{N}$.

Q36. Show by using induction that $2^n > n^2,\ n \geq 5$.

Q37. Using the induction, prove that $2^n > 2n + 1$ for all $n > 2$.

Q38. Prove that $(2n+7) < (n+3)^2$.

Q39. Prove that $1^2 + 2^2 + 3^2 + ... + n^2 > \frac{n^3}{3},\ n \in \text{N}$.

Q40. Prove that $1 + 2 + 3 + ... + n < \frac{1}{8}(2n+1)^2$.

Q41. Prove the rule of exponents $(ab)^n = a^n b^n$.

Q42. Prove that $(1+x)^n \geq (1+nx)$, for all natural numbers *n*, where $x > -1$.

Q43. Prove that the sum of first n (a) odd natural numbers is n^2 and,
(b) even natural numbers is $n^2 + n$.

Q44. By using the principle of mathematical induction prove that
$3.6 + 6.9 + 9.12 + ... + 3n(3n+3) = 3n(n+1)(n+2)$, for all $n \in \text{N}$.

Q45. Prove that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)...\left(1-\frac{1}{n+1}\right) = \frac{1}{n+1}$ by using induction.

Q46. Prove that 3^{2n} when divided by 8 leaves the remainder 1.

Q47. Show that $4^n + 15n - 1$ is divisible by 9.

Q48. Show that $x^{2n-1} - 1$ is divisible by $x - 1,\ x \neq 1$.

Q49. Prove that: $a + (a+d) + (a+2d) + ... + [a + (n-1)d] = \frac{n}{2}\left[2a + (n-1)d\right]$.

Q50. Show that $3\text{x} + 6\text{x} + 9\text{x} + ...$ upto n terms $= \frac{3}{2}n(n+1)\text{x}$ by using induction.

Q51. Prove that $\text{a}^{2\text{n}} - 1$ is divisible by (a – 1).

Q52. Prove that $\frac{\text{n}^5}{5} + \frac{\text{n}^3}{3} + \frac{7\text{n}}{15}$ is a natural number for all $\text{n} \in \text{N}$.

Any queries and/or suggestion(s), please write to me at
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Please mention your details : Name, Student/Teacher/Tutor, School/Institution, Place, Contact No. (if you wish)