Complete Study Guide & Notes On
TRIGONOMETRIC FUNCTIONS

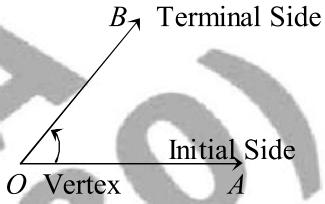
A Formulae Guide By OP Gupta (Indira Award Winner)

Mathematics is a great motivator for the entire human beings!
Its career starts with zero and goes to infinity!!

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Angle in geometry: An angle is a figure formed by two rays having common vertex called as *origin*. The rays are called sides of the angle. The measure of the angle is the amount of rotation from the direction of one ray of the angle to the other. The initial and final positions of the revolving ray are respectively called the ***initial side*** and ***terminal side*** and the revolving line is called the ***generating line*** or the ***radius vector***.

In the adjacent figure, the ray OA is the initial side and ray OB is the terminal side. And they form **angle AOB** at the vertex O.



This angle is denoted by $\angle \text{AOB}$.

With each angle a number is associated and this number is called ***measure of the angle***. There are several units for measuring this angle and we shall study about them.

◆ *In geometry an angle always lies between 0^o and 360^o and **negative angle** has no meaning.*

02. Angle in trigonometry: The idea of angle is more general in trigonometry. It may be positive or negative and of any magnitude. We know that angles in geometry are confined only till 360^o which corresponds to one complete revolution by a wheel *say*. So it is quite obvious that the angle covered in two complete revolutions is of 720^o measure and in a quarter of revolution it is of 90^o and so on.

03. Units of measurement of angles: In geometry angles are measured in terms of right angle. In order to measure smaller angles we introduce smaller units of angle. These are ***Sexagesimal or British System (Degree Measure), Centesimal or French System (Grade Measure)*** *and* ***Radian or Circular System***. Here we shall confine ourselves only to Degree measure and Radian measure. *Though you can expect a discussion about the Grade measure too in the class!*

Degree Measure: In this system of measurement a right angle is divided into 90 equal parts which are called as **degrees. Each part is equal to one degree.** Each degree is then divided in 60 equal parts called **minutes** and each minute is further divided into 60 equal parts called **seconds.** A degree, a minute and a second are denoted by the symbols 1^o, $1'$ and $1''$ respectively.

Thus, 1 Right angle $= 90^o$,

$1^o = 60'$ and $1' = 60''$.

Radian Measure: The angle subtended at the centre of circle by an arc whose length is equal to its radius is called **a radian** and is denoted by 1^C.

As shown in adjacent figure fig.A, the centre of circle is O and its radius is of *l* units. So if the length of arc *AB=l* units then, by the definition of radian given above, we have,

$\angle \text{AOB} = 1\,\text{Radian}$.

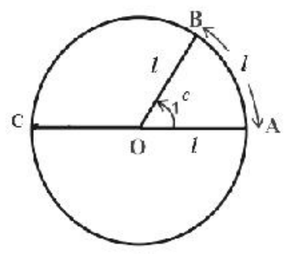


fig.A

Radian is a constant angle Consider the fig.A shown on the previous page. Let ABC be a semi-circle whose centre is at O and radius *l*. Let length of arc AB be equal to *l*. Then by definition, $\angle \text{AOB} = 1^C$.

Now produce AO and let it meet the circle at C. Then AC is a diameter of circle and arc ABC is equal to half the circumference of the circle and $\angle \text{AOC} = 2\,\text{Right angles} = 180^o$.

By our geometrical knowledge, we know that the angles subtended at the centre of a circle are proportional to the length of arc which subtends them.

i.e., $$\frac{\angle \text{AOB}}{\angle \text{AOC}} = \frac{\text{arc AB}}{\text{arc ABC}} \quad \ldots \text{(i)}$$

$\Rightarrow \quad \dfrac{1^C}{180^o} = \dfrac{l}{\pi l}$ [As *arc* ABC is a semicircle

$\Rightarrow \quad 1^C = \dfrac{180^o}{\pi}$ … (ii)

$\Rightarrow \quad 1^C = \dfrac{2 \text{ Right angles}}{\pi} = \text{Constant}.$

Understanding the π :

The π is not a whole number, nor it can be expressed in the form of a fraction, and hence not in the form of a decimal fraction, terminating or recurring. The number π has a value which can't be exactly expressed as the ratio of two whole numbers. Its value correct to 8 places of decimals, is $\pi = 3.14159265...$

In fact, the fraction $\dfrac{22}{7} = 3.14285...$ gives the value of π correct to the 2 places of decimals.

◆ *Consequently we deduce that,* $\pi^C = 180^o$ *i.e.,* π Radians $= 180^o$.

◆ *Also by (i) it can be easily deduced that* $\theta = \dfrac{l}{r}$, *if length of any arbitrary arc AC = l which subtends an angle of* θ *radians at the centre O of the circle of radius r. So,* θ (*in radian measure*) $= \dfrac{l}{r}$.

Relations in Different Measures of Angle

◆ $\text{Angle in Radian Measure} = (\text{Angle in Degree Measure}) \times \dfrac{\pi}{180}$

◆ $\text{Angle in Degree Measure} = (\text{Angle in Radian Measure}) \times \dfrac{180}{\pi}$, where $\pi = 22/7$.

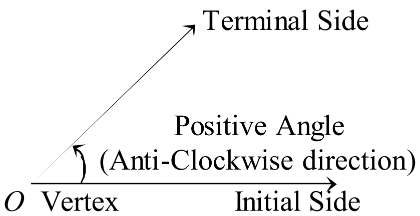
◆ *Following table can be consulted for a few frequently used standard angles:*

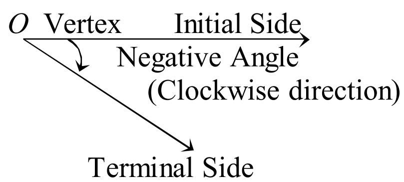
Angles in Degree	0^o	30^o	45^o	60^o	90^o	180^o	270^o	360^o
Angles in Radian	0^c	$\left(\frac{\pi}{6}\right)^c$	$\left(\frac{\pi}{4}\right)^c$	$\left(\frac{\pi}{3}\right)^c$	$\left(\frac{\pi}{2}\right)^c$	$(\pi)^c$	$\left(\frac{3\pi}{2}\right)^c$	$(2\pi)^c$

*In actual practice, we omit the exponent '**c**' and instead of writing* π^c *we simply write* π *and similarly for other angle!*

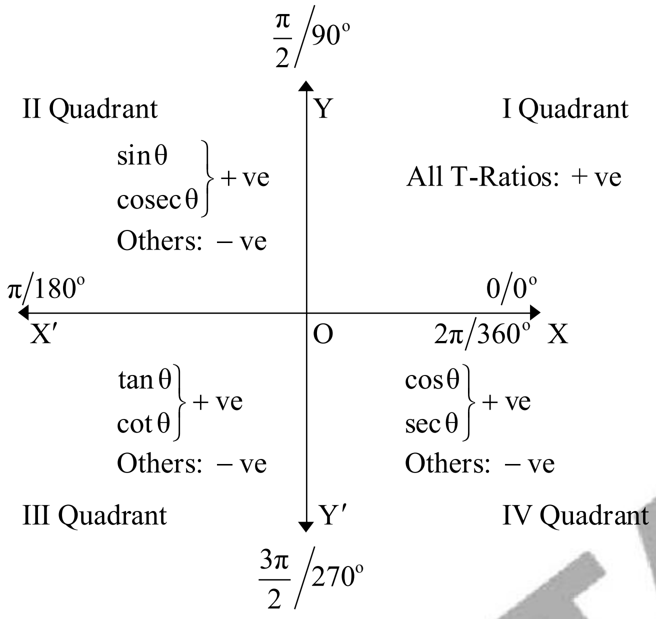
- 1 Radian=$57^o17'45''$ or 206265 seconds
- $1^o = \dfrac{\pi}{180} = 0.01745$ radians (approximately)

04. Sign of angles and Quadrants: An angle formed by **anticlockwise rotation** of the radius vector is taken as ***positive*** whereas the angle formed by **clockwise rotation** of the radius vector is taken as ***negative***. For the clarification, have a look at the figures given below:





Consider XOX' and YOY' be two mutually perpendicular lines in a plane and OX be the initial half line. The whole plane is divided into four different regions namely XOY, YOX', $X'OY'$ and XOY'. These regions are called quadrants and are respectively called 1st, 2nd, 3rd and 4th quadrants. The angle is said to be in any of these quadrants according as the terminal side lies in whichever quadrants. If the terminal side coincides with one of the axes then the angle is said to be a *quadrant angle*.



If there is any angle θ which is not a quadrant angle and radius vector rotates in the anticlockwise direction in such a way that number of revolution doesn't exceed one, we have:

- $0^o < \theta < 90^o$ If θ lies in I quadrant
- $90^o < \theta < 180^o$ If θ lies in II quadrant
- $180^o < \theta < 270^o$ If θ lies in III quadrant
- $270^o < \theta < 360^o$ If θ lies in IV quadrant

Also when terminal side coincides with OY, $\theta = 90^o$
when terminal side coincides with OX′, $\theta = 180^o$
when terminal side coincides with OY′, $\theta = 270^o$
when terminal side coincides with OX, $\theta = 360^o$.

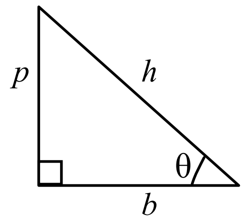
◆ *Following table will be sufficient to give you an idea about the discussion we just have had:*

Angles (→) / **T- Ratios** (↓)	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$ **OR** $-\theta$	$2\pi+\theta$
sin	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$
cos	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$	$\cos\theta$
tan	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$
cot	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$
sec	$\text{cosec}\theta$	$-\text{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\text{cosec}\theta$	$\text{cosec}\theta$	$\sec\theta$	$\sec\theta$
cosec	$\sec\theta$	$\sec\theta$	$\text{cosec}\theta$	$-\text{cosec}\theta$	$-\sec\theta$	$-\sec\theta$	$-\text{cosec}\theta$	$\text{cosec}\theta$

05. Recapitulation of previous class: Following is a list of those relations which you have studied in your last class.

Please note that their proof has not been mentioned here. Though you can anytime discuss it with me again in case you have forgotten.

◆ *Trigonometric ratios and sides of a right angled triangle:*



- $\sin\theta = \dfrac{p}{h}$
- $\cos\theta = \dfrac{b}{h}$
- $\tan\theta = \dfrac{p}{b}$
- $\text{cosec}\theta = \dfrac{h}{p}$
- $\sec\theta = \dfrac{h}{b}$
- $\cot\theta = \dfrac{b}{p}$

◆ *Trigonometric Identities:*

- $\sin^2\theta + \cos^2\theta = 1$
- $1 + \tan^2\theta = \sec^2\theta$
- $1 + \cot^2\theta = \text{cosec}^2\theta$

◆ *Relation between trigonometric ratios:*

- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$
- $\tan\theta = \dfrac{1}{\cot\theta}$
- $\tan\theta.\cot\theta = 1$
- $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$
- $\text{cosec}\,\theta = \dfrac{1}{\sin\theta}$
- $\sec\theta = \dfrac{1}{\cos\theta}$.

◆ *Following table includes trigonometric ratio of standard angles:*

Degree / Radian	0°	30°	45°	60°	90°
T – Ratios	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
sin	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
cos	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
tan	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
cosec	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
sec	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
cot	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

◆ *Following table demonstrates the domain and range of trigonometric functions:*

T- Functions	**Domain**	**Range**
$\sin x$	R	$[-1, 1]$
$\cos x$	R	$[-1, 1]$
$\tan x$	$\{x \in R : x \neq (2n+1)\pi/2, n \in Z\}$	R
$\cot x$	$\{x \in R : x \neq n\pi, n \in Z\}$	R
$\text{cosec}\, x$	$\{x \in R : x \neq n\pi, n \in Z\}$	$R - (-1, 1)$
$\sec x$	$\{x \in R : x \neq (2n+1)\pi/2, n \in Z\}$	$R - (-1, 1)$

06. Some useful trigonometric identities and formulae:

Trigonometric identities

a) $\sin^2\theta+\cos^2\theta=1$

b) $1+\tan^2\theta=\sec^2\theta$

c) $1+\cot^2\theta=\text{cosec}^2\theta$

Addition / subtraction formulae & some related results

a) $\sin(A\pm B)=\sin A\cos B\pm\cos A\sin B$

b) $\cos(A\pm B)=\cos A\cos B\mp\sin A\sin B$

c) $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

d) $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

e) $\tan(A\pm B)=\dfrac{\tan A\pm\tan B}{1\mp\tan A\tan B}$

f) $\cot(A\pm B)=\dfrac{\cot B\cot A\mp 1}{\cot B\pm\cot A}$

Transformation of sums / differences into products & vice-versa

a) $\sin C+\sin D=2\sin\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

b) $\sin C-\sin D=2\cos\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

c) $\cos C+\cos D=2\cos\dfrac{C+D}{2}\cos\dfrac{C-D}{2}$

d) $\cos C-\cos D=-2\sin\dfrac{C+D}{2}\sin\dfrac{C-D}{2}$

e) $2\sin A\cos B=\sin(A+B)+\sin(A-B)$

f) $2\cos A\sin B=\sin(A+B)-\sin(A-B)$

g) $2\cos A\cos B=\cos(A+B)+\cos(A-B)$

h) $2\sin A\sin B=\cos(A-B)-\cos(A+B)$

Multiple angle formulae involving 2A and 3A

a) $\sin 2A=2\sin A\cos A$

b) $\sin A=2\sin\dfrac{A}{2}\cos\dfrac{A}{2}$

c) $\cos 2A=\cos^2 A-\sin^2 A$

d) $\cos A=\cos^2\dfrac{A}{2}-\sin^2\dfrac{A}{2}$

e) $\cos 2A=2\cos^2 A-1$

f) $2\cos^2 A=1+\cos 2A$

g) $\cos 2A=1-2\sin^2 A$

h) $2\sin^2 A=1-\cos 2A$

i) $\sin 2A=\dfrac{2\tan A}{1+\tan^2 A}$

j) $\cos 2A=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

k) $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

l) $\sin 3A=3\sin A-4\sin^3 A$

m) $\cos 3A=4\cos^3 A-3\cos A$

n) $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

07. Trigonometric equations, General solutions and Principal solutions: An equation involving one or more trigonometric ratios of unknown angle is called a trigonometric equation. It is important to note that a **trigonometric identity is satisfied for every value of the unknown angle where as trigonometric equation is satisfied for some values (finite or infinite) of unknown angle**.

Since trigonometric functions are periodic functions, therefore, solutions of trigonometric equations can be generalized with the help of **periodicity of trigonometric functions**. The solution consisting of all possible solutions of a trigonometric equation is called its general solution.

General solution of trigonometric equation of following types

a) $\sin x=0$ gives $x=n\pi$, where $n\in Z$

b) $\cos x=0$ gives $x=(2n+1)\dfrac{\pi}{2}$, where $n\in Z$

c) $\tan x=0$ gives $x=n\pi$, where $n\in Z$

d) $\sin x=\sin y$ gives $x=n\pi+(-1)^n y$, where $n\in Z$

e) $\cos x=\cos y$ gives $x=2n\pi\pm y$, where $n\in Z$

f) $\tan x=\tan y$ gives $x=n\pi+y$, where $n\in Z$.

Principal solution: The solution of a trigonometric equation in x for which $0\le x<2\pi$ are called the principal solutions.

◆ *Did you know?*

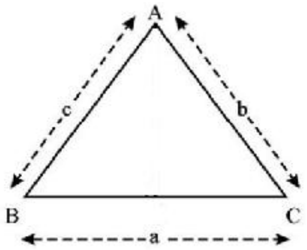
$\oint$ $\sin\theta = (-1)^n$ *if* $\theta = (2n+1)\dfrac{\pi}{2},\ n \in Z$

$\oint$ $\cos\theta = (-1)^n$ *if* $\theta = n\pi,\ n \in Z$

$\oint$ $\sin\theta = 0$ *if* $\theta = n\pi,\ n \in Z$

$\oint$ $\cos\theta = 0$ *if* $\theta = (2n+1)\dfrac{\pi}{2},\ n \in Z$.

08. Law of sines: The sine rule states that the lengths of the sides of a triangle are proportional to the sines of angles opposite to them *i.e.,* in ΔABC, we have $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$.



◆ $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = k\ (say) \Rightarrow a = k\sin A,\ b = k\sin B,\ k = c\sin C$.

◆ $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c} = \lambda\ (say) \Rightarrow \sin A = a\lambda,\ \sin B = b\lambda,\ \sin C = c\lambda$.

09. Law of cosines: In any ΔABC, we have

a) $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$ or $a^2 = b^2 + c^2 - 2bc\cos A$

b) $\cos B = \dfrac{c^2 + a^2 - b^2}{2ca}$ or $b^2 = c^2 + a^2 - 2ca\cos B$

c) $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab}$ or $c^2 = a^2 + b^2 - 2ab\cos C$.

Any queries and/or suggestion(s), please write to me at
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Please mention your details : Name, Student/Teacher/Tutor, School/Institution, Place, Contact No. (if you wish)