Complete Study Guide & Notes On
RELATIONS & FUNCTIONS

A Formulae Guide By OP Gupta (Indira Award Winner)

If doing Maths is not fun, then why do it?

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Ordered pair and Equality of ordered pairs: A pair of objects written in a particular order is called an ordered pair. An ordered pair is written by listing its two members in a particular order, separated by a comma and enclosing the pair in parenthesis. In the ordered pair (*a, b*), *a* is called the first component or the first element and *b* is called the second component or second element. In an ordered pair (*a, b*), the order in which the elements *a* and *b* appear in the bracket is important. The ordered pairs (0, 1) and (1, 0) though consists of the same numbers 0 and 1, but are different as they appear in different order.

◆ *Note that the ordered pair (a, b) is different from the set {a, b}.*

Equality of ordered pairs: Two ordered pairs (a_1, b_1) and (a_2, b_2) are said to be equal iff $a_1 = a_2$ and $b_1 = b_2$. In other words, if the ordered pairs (a_1, b_1) and (a_2, b_2) are equal, we write $(a_1, b_1) = (a_2, b_2)$.

Thus, $(a_1, b_1) = (a_2, b_2) \Rightarrow a_1 = a_2$ and $b_1 = b_2$.

e.g. $(1, 2) = (1, 2)$ but $(1, 2) \neq (2, 1)$.

02. Cartesian product of sets: Given two non-empty sets A and B, the Cartesian product $A \times B$ is the set of all ordered pairs of elements from A and B, *i.e.*, $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. Also $A \times B$ is read as 'A cross B' or 'product set of A and B'.

Thus $(a, b) \in A \times B \Leftrightarrow a \in A$ and $b \in B$.

◆ To enlist the elements of $A \times B$, we first choose the first element of A, then form all the ordered pairs with it as first component and elements of B as second component separated by comma. Next we choose the second element of A; and again form the ordered pairs in the same way and so on.

e.g. Let A={1, 2}, B={*a, b, c*}. Then $A \times B = \{1, 2\} \times \{a, b, c\} = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}$.

◆ Note the followings:

a) If $A = \phi$ or $B = \phi$ *i.e., if at least one of A and B is empty set, then* $A \times B = \phi$.

b) $A \times B \neq \phi \Leftrightarrow A \neq \phi$ *and* $B \neq \phi$.

c) $A \times B$ *may or may not be equal to* $B \times A$.

d) $A \times A$ *is also denoted by* A^2.

e) If at least one of A and B is an infinite set and other is a non-empty set, then $A \times B$ *is an infinite set.*

f) $n(A \times B) = n(A) \cdot n(B)$ *i.e., if there are p elements in A and q elements in B then, the* ***number of elements in*** $A \times B$ ***will be pq.***

g) $A \times A \times A = \{(a, b, c);\ a, b, c \in A\}$. *Here* (a, b, c) *is called an* ***ordered triplet***.

03. Relations: A relation R, from a non-empty set A to another non-empty set B is mathematically defined as an arbitrary subset of $A \times B$. Equivalently, any subset of $A \times B$ is a relation from A to B.

Thus, R is a relation from A to B $\Leftrightarrow R \subseteq A \times B$

$$\Leftrightarrow R \subseteq \{(a, b) : a \in A, b \in B\}.$$

Illustrations:

a) Let $A = \{1,2,4\}$, $B = \{4,6\}$. *Let* $R = \{(1,4),(1,6),(2,4),(2,6),(4,6)\}$. *Here* $R \subseteq A \times B$ *and therefore R is a relation from A to B.*

b) Let $A=\{1,2,3\}$, $B=\{2,3,5,7\}$. *Let* $R=\{(2,3),(3,5),(5,7)\}$. *Here* $R\not\subset A\times B$ *and therefore R is not a relation from A to B. Since* $(5,7)\in R$ *but* $(5,7)\notin A\times B$.

c) Let $A=\{-1,1,2\}$, $B=\{1,4,9,10\}$. *Let aRb means* $a^2=b$ *then,* $R=\{(-1,1),(1,1),(2,4)\}$.

◆ Note the followings:

a) A relation from A to B is also called a relation from A into B.

b) $(a,b)\in R$ *is also written as aRb (read as* ***a is R related to b****).*

c) Let A and B be two non-empty finite sets having p and q elements respectively. Then $n(A\times B)=n(A).n(B)=pq$. *Then total number of subsets of* $A\times B=2^{pq}$. *Since each subset of* $A\times B$ *is a relation from A to B, therefore* ***total number of relations from A to B is*** $\mathbf{2^{pq}}$.

d) A relation R from A to A is also stated as a relation on A. Remember that this kind of relation is called as ***binary relation*** *as well. Hence every subset of* $A\times A$ *is called a binary relation on A.*

04. Domain and range of a relation:

a) Domain of a relation: Let R be a relation from A to B. The domain of relation R is the set of all those elements $a\in \mathrm{A}$ such that $(a,b)\in \mathrm{R}$ for some $b\in \mathrm{B}$. Domain of R is precisely written as **Dom.(R)** symbollically.

Thus, $\mathrm{Dom.(R)}=\{a\in \mathrm{A}:(a,b)\in \mathrm{R} \text{ for some } b\in \mathrm{B}\}$.

That is, domain of R is the **set of first component of all the ordered pairs** which belong to R.

b) Range of a relation: Let R be a relation from A to B. The range of relation R is the set of all those elements $b\in \mathrm{B}$ such that $(a,b)\in \mathrm{R}$ for some $a\in \mathrm{A}$.

Thus, Range of $\mathrm{R}=\{b\in \mathrm{B}:(a,b)\in \mathrm{R} \text{ for some } a\in \mathrm{A}\}$.

That is, range of R is the **set of second components of all the ordered pairs** which belong to R.

c) Codomain of a relation: Let R be a relation from A to B. Then B is called the co-domain of the relation R. So we can observe that **co-domain of a relation R from A into B is the set B as a whole**.

Thus, **Range** $\subseteq$ **Codomain** .

Illustrations:

a) Let $A=\{1,2,3,7\}$, $B=\{3,6\}$. *Let aRb means* $a<b$. *Then* $R=\{(1,3),(1,6),(2,3),(2,6),(3,6)\}$. *Here we have,* $Dom.(R)=\{1,2,3\}$, *Range of* $R=\{3,6\}$, *Codomain of* $R=B=\{3,6\}$.

b) Let $A=\{1,2,3\}$, $B=\{2,4,6,8\}$. *Let* $R_1=\{(1,2),(2,4),(3,6)\}$, $R_2=\{(2,4),(2,6),(3,8),(1,6)\}$. *Then both* R_1 *and* R_2 *are relations from A to B because* $R_1\subseteq A\times B$, $R_2\subseteq A\times B$. *Here* $Dom.(R_1)=\{1,2,3\}$, *Range of* $R_1=\{2,4,6\}$; $Dom.(R_2)=\{2,3,1\}$, *Range of* $R_2=\{4,6,8\}$.

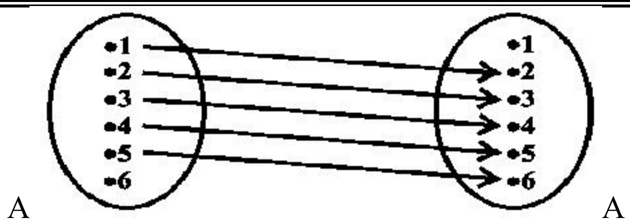
05. Representation of a relation: A relation from a set A to set B can be represented in any one of the following forms–

a) Roster form: In this form, a relation R is represented by the set of all ordered pairs belonging to R. *e.g.* Let A={–1, 1, 2} and B={1, 4, 9, 10}. Let $a\,\mathrm{R}\,b$ means $a^2=b$. Then R(in roster form) = {(–1, 1), (1, 1), (2, 4)}.

b) Set-builder form: In this form, the relation R is represented as {(*a, b*): $a\in \mathrm{A}$, $b\in \mathrm{B}$, *a*...*b*}, the blank is to be replaced by the rule which associates *a* and *b*.

e.g. Let A={1, 3, 5, 7}, B={2, 4, 6, 8}. Let R={(1, 2), (3, 4), (5, 6), (7, 8)} then R in the set-builder form can be written as R={(*a, b*): $a\in \mathrm{A}$, $b\in \mathrm{B}$, $a-b=-1$}.

c) Arrow diagram: In this form, the relation R is represented by drawing arrows from first components to the second components of all the ordered pairs belonging to R.



e.g. Let A={1, 2, 3, 4, 5, 6} and R be the relation {(*x*, *y*): *y*=*x*+1} from A to A, then R={(1, 2), (2, 3), (3, 4), (4, 5), (5, 6)}. This relation R from A to A can be represented by the arrow diagram as shown in the figure given above.

06. Constant and types of variables:

a) Constant: A constant is a symbol which retains the same value throughout a set of operations. So, a symbol which denotes a particular number is a constant. Constants are usually denoted by the symbols *a, b, c, k, l, m,...* etc.

b) Variable: It is a symbol which takes a number of values *i.e.,* it can take any arbitrary values over the interval on which it has been defined. For example if x is a variable over R (set of real numbers) then we mean that x can denote any arbitrary real number. Variables are usually denoted by the symbols *x, y, z, u, v,...* etc.

c) Independent variable: That variable which can take any arbitrary value from a given set is termed as an independent variable.

d) Dependent variable: That variable whose value depends on the independent variable is called a dependent variable.

07. Defining a function: Consider A and B be two non- empty sets then, a *rule* f *which associates each element of A with a unique element of B is called* ***a function*** or the ***mapping from A to B*** or ***f maps A to B***. If f is a mapping from A to B then, we write $f: \text{A} \to \text{B}$ which is read as ***'f is a mapping from A to B'*** or ***'f is a function from A to B'***.
If f associates $a \in \text{A}$ to $b \in \text{B}$, then we say that **'*b* is the image of the element *a* under the function** f**'** or **'*b* is the** f**–image of *a*'** or **'the value of** f **at *a*'** and denote it by $f(a)$ and we write $b = f(a)$. The ***element a is called the pre-image*** **or** ***inverse-image of b***.

◆ Thus for a function from A to B,

a) A and B should be non-empty.

b) each element of A should have image in B.

c) no element of A should have more than one image in B.

> *A relation from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.*

◆ *Note that if A and B have respectively m and n number of elements then the* ***number of functions defined from A to B is given as*** n^m.

08. Domain, Co-domain and Range of a function: The set A is called the domain of the function f and the set B is called the co- domain. The set of the images of all the elements of A under the function f is called the range of the function f and is denoted as $f(\text{A})$. Thus range of the function f is $f(\text{A}) = \{f(x): x \in \text{A}\}$. Clearly $f(\text{A}) \subseteq \text{B}$.

◆ Note the followings:

a) It is necessary that every f-image is in B; but there may be some elements in B which are not f-images of any element of A i.e. whose pre-image under f is not in A.

b) Two or more elements of A may have same image in B.

c) $f: x \to y$ *means that under the function f from A to B, an element x of A has image y in B.*

d) Usually we denote the function f by writing $y = f(x)$ *and read it as y is a function of x.*

09. Points to remember for finding the domain and range of a function:

Domain: If a function is expressed in the form $y = f(x)$, then domain of f means **set of all those real values of *x* for which *y* is real (*i.e.*, *y* is well- defined)**.

- Remember the following points while finding the domain of a function:
  - *i) Negative number should not occur under square root (even root) i.e., expression under the square root sign must be non-negative i.e., the expression must be ≥ 0.*
  - *ii) Denominator should never be zero.*
  - *iii) For $\log_b a$ to be defined $a > 0, b > 0$ and $b \neq 1$. Also note that $\log_b 1$ is equal to zero i.e. 0.*

> *A function which has either R (set of real numbers) or one of its subsets as its range is called a **real valued function**. Further, if it domain is also either R or a subset of R, it is called a real function. In other words, if the domain and range of a function f are subsets of R, then f is said to be a **real valued function of a real variable** or a **real function**.*

Range: If a function is expressed in the form $y = f(x)$, then range of f means **set of all possible real values of *y* corresponding to every value of *x* in its domain.**

- Remember the following points:
  - ***i)*** *Firstly find the domain of the given function.*
  - ***ii)*** *If the domain does not contain an interval, then find the values of y putting these values of x from the domain. The set of all these values of y obtained will be the range.*
  - ***iii)*** *If domain is the set of all real nos. R or set of all real nos. except a few points, then express x in terms of y and from this find real values of y for which x is real and belongs to domain.*

10. Types of intervals:

a) Open interval: If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying strictly between a and b is called an open interval. It is denoted by $]a,b[$ or (a,b) i.e., $\{x \in \mathrm{R} : a < x < b\}$.

b) Closed interval: If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes both a and b as well is known as a closed interval. It is denoted by $[a,b]$ *i.e.*, $\{x \in \mathrm{R} : a \leq x \leq b\}$.

c) Open Closed interval: If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it excludes a and includes only b is known as an open closed interval. It is denoted by $]a,b]$ or $(a,b]$ *i.e.*, $\{x \in \mathrm{R} : a < x \leq b\}$.

d) Closed Open interval: If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes only a and excludes b is known as a closed open interval. It is denoted by $[a,b[$ *or* $[a,b)$ *i.e.* $\{x \in \mathrm{R} : a \leq x < b\}$.

11. Algebra of real functions:

a) Addition of two real functions: $(f+g)(x) = f(x) + g(x)$, for all $x \in \mathrm{R}$.

b) Subtraction of real function from another: $(f-g)(x) = f(x) - g(x)$, for all $x \in \mathrm{R}$.

c) Multiplication by a scalar: $(\alpha f)(x) = \alpha f(x)$, for all $x \in \mathrm{R}$.

d) Multiplication of two real function: $(f \cdot g)(x) = f(x)g(x)$, for all $x \in \mathrm{R}$.

e) Quotient of two real functions: $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$, for all $x \in \mathrm{R}$.

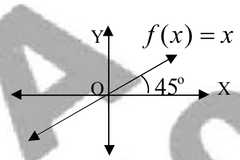
12. Some important real functions and their domain & range:

FUNCTION	REPRESENTATION	DOMAIN	RANGE
a) Identity function	$I(x) = x \ \forall\, x \in \mathrm{R}$	R	R
b) Absolute value function or Modulus function	$f(x) = \lvert x \rvert = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$	R	$[0,\infty)$

c) Greatest integer function or integral function or step function	$f(x)=[x]$ or $f(x)=\lfloor x\rfloor\ \forall x\in \mathrm{R}$	R	Z
d) Smallest integer function	$f(x)=\lceil x\rceil\ \forall x\in \mathrm{R}$	R	Z
e) Signum function	$f(x)=\begin{cases}\dfrac{\lvert x\rvert}{x}\ \text{if}\ x\neq 0\\ 0\ \text{if}\ x=0\end{cases}=\begin{cases}1\ \text{if}\ x>0\\ 0\ \text{if}\ x=0\\ -1\ \text{if}\ x<0\end{cases}$	R	$\{-1,0,1\}$
f) Exponential function	$f(x)=a^x\ \forall\ a\neq 1,\ a>0$	R	$(0,\infty)$
g) Logarithmic function	$f(x)=\log_a x\ \forall\ a\neq 1,\ a>0$ and $x>0$	$(0,\infty)$	R

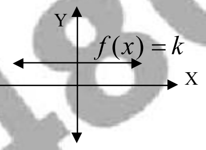
13. Some functions and their graphs:

(i) ***Identity function***



$f(x)=x$ for all $x\in \mathrm{R}$
Domain = R
Range = R.

(ii) ***Constant function***

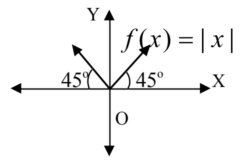


$f(x)=k$ where k is a fixed real number
Domain = R
Range = $\{k\}$.

(iii) ***Polynomial function***

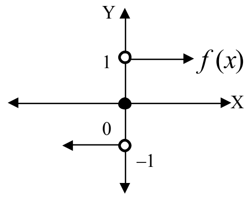
$f(x)=a_0+a_1x+a_2x^2+a_3x^3+...+a_nx^n$ where $a_0,a_1,a_2,a_3,....,a_n\in \mathrm{R}$ and n is a non-negative integer.
Domain = R
Range = It depends on the polynomial.

(iv) ***Modulus function (Absolute value function)***



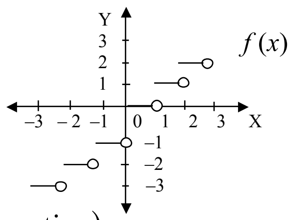
$f(x)=|x|=\begin{cases}x,\ \text{if}\ x\geq 0\\ -x,\ \text{if}\ x<0\end{cases}$
Domain = R
Range = Non negative real numbers = $[0,\infty)$.

(v) ***Signum function***



$f(x)=\begin{cases}\dfrac{|x|}{x},\ \text{if}\ x\neq 0\\ 0,\ \text{if}\ x=0\end{cases}$ or $f(x)=\begin{cases}1,\ \text{if}\ x>0\\ 0,\ \text{if}\ x=0\\ -1,\ \text{if}\ x<0\end{cases}$
Domain = R
Range = $\{-1,\ 0,\ 1\}$.

(vi) ***Greatest integer function (Step function)***



$f(x)=[x]$ or $\lfloor x\rfloor,\ x\in \mathrm{R}$
Domain = R

Range = Z, all integers $(i.e.,$ + ve or – ve or non-positive non-negative$)$.