CBSE 2013 DELHI COMPARTMENT EXAMINATION
[Solutions With Detailed Explanations – Set 2]

Time allowed: 3 Hours **Maximum Marks: 100**

SECTION – A
[Question numbers 01 to 10 carry 1 mark each.]

Q01. The total cost associated with provision of free mid-day meals to x students of a school in primary classes is given by

$$C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$$

If the marginal cost is given by rate of change $\frac{dC}{dx}$ of total cost, write the marginal cost of food for 300 students. What value is shown here?

Sol. We have $C(x) = 0.005x^3 - 0.02x^2 + 30x + 50$ $\Rightarrow C'(x) = 0.015x^2 - 0.04x + 30 + 0$

$\therefore C'(300) = 0.015(300)^2 - 0.04(300) + 30 = 1368$.

So the marginal cost of food for 300 students is ₹1368.

Value: The mid-day program run by Govt. is the largest such program across the world. It is beneficial for the students of young age to provide them with well nourished meal in the school premises along with the education.

Q02. If $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$, then write the value of (x + y).

Sol. We have $2\begin{bmatrix}1 & 3\\0 & x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$ $\Rightarrow \begin{bmatrix}2 & 6\\0 & 2x\end{bmatrix}+\begin{bmatrix}y & 0\\1 & 2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$

$\Rightarrow \begin{bmatrix}2+y & 6\\1 & 2x+2\end{bmatrix}=\begin{bmatrix}5 & 6\\1 & 8\end{bmatrix}$ $\therefore 2 + y = 5,\ 2x + 2 = 8$ [By equality of matrices

i.e., $y = 3,\ x = 3$ $\therefore x + y = 6$.

Q03. If $\begin{vmatrix}2x & x+3\\2(x+1) & x+1\end{vmatrix}=\begin{vmatrix}1 & 5\\3 & 3\end{vmatrix}$, then write the value of x.

Sol. Given $\begin{vmatrix}2x & x+3\\2(x+1) & x+1\end{vmatrix}=\begin{vmatrix}1 & 5\\3 & 3\end{vmatrix}$ $\Rightarrow (2x)(x+1) - 2(x+1)(x+3) = 3 - 3\times 5$

On solving, we get: $x = 1$.

Q04. If A is a square matrix of order 3 such that |adj A| = 64, find the value of |A|.

Sol. Since $|\text{adj } A| = |A|^{n-1}$, where n is order of matrix A.

So, $|A|^{3-1} = 64$ $\Rightarrow |A|^2 = 64$ $\Rightarrow |A| = \pm 8$.

Q05. Write the value of the following :

$$\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right).$$

Sol. Given $\tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{a+b}\right) = \tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a-b}{b+a}\right)$

$$\Rightarrow \quad = \tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{\frac{a}{b}-1}{1+1\times\left(\frac{a}{b}\right)}\right) = \tan^{-1}\left(\frac{a}{b}\right)-\tan^{-1}\left(\frac{a}{b}\right)+\tan^{-1}(1)$$

$$\Rightarrow \quad = \frac{\pi}{4}.$$ [Using $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y$

Q06. Write the principal value of $\left[\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\left(-\frac{1}{2}\right)\right]$.

Sol. We have $\left[\cos^{-1}\frac{\sqrt{3}}{2}+\cos^{-1}\left(-\frac{1}{2}\right)\right]=\left[\cos^{-1}\cos\frac{\pi}{6}+\cos^{-1}\cos\frac{2\pi}{3}\right]$

$\Rightarrow \quad =\frac{\pi}{6}+\frac{2\pi}{3}=\frac{5\pi}{6}$ {∵ Range of principal value branch of $\cos^{-1}x$ is $[0,\pi]$

Q07. Find the direction cosines of the line $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$.

Sol. Given line is $\frac{4-x}{2}=\frac{y}{6}=\frac{1-z}{3}$ $\Rightarrow \frac{x-4}{-2}=\frac{y-0}{6}=\frac{z-1}{-3}$

The direction ratios of line are –2, 6, –3.

So, its d. c.'s are given as $\pm\frac{-2}{\sqrt{(-2)^2+(6)^2+(-3)^2}},\pm\frac{6}{\sqrt{4+36+9}},\pm\frac{-3}{7}$ i.e., $\mp\frac{2}{7},\pm\frac{6}{7},\mp\frac{3}{7}$.

Q08. Write the projection of the vector $7\hat{i}+\hat{j}-4\hat{k}$ on the vector $2\hat{i}+6\hat{j}+3\hat{k}$.

Sol. Let $\vec{a}=7\hat{i}+\hat{j}-4\hat{k}$ and $\vec{b}=2\hat{i}+6\hat{j}+3\hat{k}$.

So, projection of $\vec{a}$ on $\vec{b}=\vec{a}\,.\,\hat{b}=(7\hat{i}+\hat{j}-4\hat{k}).\left(\frac{2\hat{i}+6\hat{j}+3\hat{k}}{\sqrt{2^2+6^2+3^2}}\right)=\frac{8}{7}$.

Q09. Write the value of λ so that the vectors $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other.

Sol. Since $\vec{a}=2\hat{i}+\lambda\hat{j}+\hat{k}$ and $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ are perpendicular to each other, so $\vec{a}.\vec{b}=0$

i.e., $(2\hat{i}+\lambda\hat{j}+\hat{k}).(\hat{i}-2\hat{j}+3\hat{k})=0$ $\Rightarrow 2-2\lambda+3=0$ $\Rightarrow \lambda=\frac{5}{2}$.

Q10. Write the degree of the differential equation :

$\left(\frac{dy}{dx}\right)^4+3y\frac{d^2y}{dx^2}=0$.

Sol. Degree : 1.

SECTION – B

[Question numbers 11 to 22 carry 4 marks each.]

Q11. If $f(x)=\begin{cases}\frac{1-\cos 4x}{x^2}, & \text{when } x<0\\ a, & \text{when } x=0\\ \frac{\sqrt{x}}{\left(\sqrt{16+\sqrt{x}}\right)-4}, & \text{when } x>0\end{cases}$ and f is continuous at x = 0, find the value of a.

Sol. Since the function f(x) is continuous at x = 0 so, LHL (at x = 0) = RHL (at x = 0) = f(0) …(i)

Now, f(0) = a …(ii)

RHL (at x = 0): $\lim_{x\to0^+}\frac{\sqrt{x}}{\left(\sqrt{16+\sqrt{x}}\right)-4}=\lim_{x\to0^+}\frac{\sqrt{x}}{\left(\sqrt{16+\sqrt{x}}\right)-4}\times\frac{\left(\sqrt{16+\sqrt{x}}\right)+4}{\left(\sqrt{16+\sqrt{x}}\right)+4}$

$$=\lim_{x\to0^+}\frac{\sqrt{x}\left[\left(\sqrt{16+\sqrt{x}}\right)+4\right]}{\left(\sqrt{16+\sqrt{x}}\right)^2-4^2}$$

$$=\lim_{x\to0^+}\frac{\sqrt{x}\left[\left(\sqrt{16+\sqrt{x}}\right)+4\right]}{16+\sqrt{x}-16}=\lim_{x\to0^+}\left[\left(\sqrt{16+\sqrt{x}}\right)+4\right]$$

$= \sqrt{16+\sqrt{0}} + 4 = 8$ …(iii)

By (i), (ii) and (iii), we get: a = 8.

Q12. Evaluate : $\int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x} dx$.

(OR) Evaluate : $\int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$.

Sol. Let $I = \int \frac{(3\sin x - 2)\cos x}{5 - \cos^2 x - 4\sin x} dx$ $\Rightarrow I = \int \frac{(3\sin x - 2)\cos x}{5 - (1-\sin^2 x) - 4\sin x} dx$

$\Rightarrow I = \int \frac{(3\sin x - 2)\cos x}{\sin^2 x - 4\sin x + 4} dx$ $\Rightarrow I = \int \frac{(3\sin x - 2)\cos x}{(\sin x - 2)^2} dx$

Put $\sin x = t \Rightarrow \cos x\, dx = dt$.

$\therefore\ I = \int \frac{(3t-2)}{(t-2)^2} dt$ …(i)

Consider $\frac{(3t-2)}{(t-2)^2} = \frac{A}{(t-2)} + \frac{B}{(t-2)^2}$ $\Rightarrow (3t-2) = A(t-2) + B$

On equating the coefficients on both sides, we get: A = 3, B = 4.

By (i), we have: $I = \int \left[\frac{3}{(t-2)} + \frac{4}{(t-2)^2}\right] dt$ $\Rightarrow I = 3\log|t-2| - \frac{4}{t-2} + k$

So, $I = 3\log|\sin x - 2| - \frac{4}{\sin x - 2} + k$.

(OR) Let $I = \int e^{2x}\left(\frac{1-\sin 2x}{1-\cos 2x}\right)dx$ $\Rightarrow I = \int e^{t}\left(\frac{1-\sin t}{1-\cos t}\right)\frac{dt}{2}$, where 2x = t, dx = dt/2

$\Rightarrow I = \frac{1}{2}\int e^t \left(\frac{1 - 2\sin\left(\frac{t}{2}\right)\cos\left(\frac{t}{2}\right)}{2\sin^2\left(\frac{t}{2}\right)}\right) dt$ $\Rightarrow I = \frac{1}{2}\int e^t \left[\frac{1}{2}\text{cosec}^2\left(\frac{t}{2}\right) - \cot\left(\frac{t}{2}\right)\right] dt$

$\Rightarrow I = \frac{1}{4}\int e^t \text{cosec}^2\left(\frac{t}{2}\right) dt - \frac{1}{2}\int e^t \cot\left(\frac{t}{2}\right) dt$ [Apply By Parts in 1st integral

$\therefore\ I = \frac{1}{4}\left[e^t \int \text{cosec}^2\left(\frac{t}{2}\right) dt - \int\left(\frac{d}{dt}(e^t)\int \text{cosec}^2\left(\frac{t}{2}\right)dt\right)dt\right] - \frac{1}{2}\int e^t \cot\left(\frac{t}{2}\right) dt$

$\Rightarrow I = \frac{1}{4}\left[-2e^t \cot\left(\frac{t}{2}\right) + 2\int e^t \cot\left(\frac{t}{2}\right) dt\right] - \frac{1}{2}\int e^t \cot\left(\frac{t}{2}\right) dt$

$\Rightarrow I = -\frac{1}{2}e^t \cot\left(\frac{t}{2}\right) + k$ $\therefore\ I = -\frac{1}{2}e^{2x}\cot x + k$.

Q13. Evaluate : $\int_0^{\pi/4} \log(1+\tan x) dx$.

Sol. Let $I = \int_0^{\pi/4} \log(1+\tan x) dx$ …(i) [By using $\int_0^a f(x)dx = \int_0^a f(a-x)dx$

$\Rightarrow I = \int_0^{\pi/4} \log\left(1 + \tan\left(\frac{\pi}{4} - x\right)\right) dx$ $\Rightarrow I = \int_0^{\pi/4} \log\left(1 + \frac{\tan\frac{\pi}{4} - \tan x}{1 + \tan\frac{\pi}{4}\tan x}\right) dx$

$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$ $\qquad \Rightarrow I = \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$ …(ii)

Adding (i) and (ii), we get: $2I = \int_0^{\pi/4} \log(1+\tan x)dx + \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$

$\Rightarrow 2I = \int_0^{\pi/4} \log\left[(1+\tan x)\left(\frac{2}{1+\tan x}\right)\right]dx \qquad \Rightarrow 2I = \int_0^{\pi/4} \log 2\, dx$

$\Rightarrow 2I = \log 2\left[x\right]_0^{\pi/4} \qquad \therefore I = \frac{\pi}{8}(\log 2)$.

Q14. The Cartesian equations of a line are 6x – 2 = 3y + 1 = 2z – 2. Find the direction cosines of the line. Write down the Cartesian and vector equations of a line passing through (2,–1,–1) which is parallel to the given line.

(OR) Find the shortest distance between the two lines whose vector equations are $\vec{r} = (6\hat{i}+2\hat{j}+2\hat{k}) + \lambda(\hat{i}-2\hat{j}+2\hat{k})$ and $\vec{r} = (-4\hat{i}-\hat{k}) + \mu(3\hat{i}-2\hat{j}-2\hat{k})$.

Sol. Given line L is: 6x – 2 = 3y + 1 = 2z – 2 $\Rightarrow \frac{x-\frac{1}{3}}{1/6} = \frac{y-\left(-\frac{1}{3}\right)}{1/3} = \frac{z-1}{1/2} \Rightarrow \frac{x-\frac{1}{3}}{1} = \frac{y+\frac{1}{3}}{2} = \frac{z-1}{3}$

So, d.r.'s of line L are : 1, 2, 3.

$\therefore$ direction cosines of line L are: $\pm\frac{1}{\sqrt{1^2+2^2+3^2}}, \pm\frac{2}{\sqrt{1+4+9}}, \pm\frac{3}{\sqrt{14}}$ i.e., $\pm\frac{1}{\sqrt{14}}, \pm\frac{2}{\sqrt{14}}, \pm\frac{3}{\sqrt{14}}$

Now, since the parallel lines have the proportional d.r.'s so, d.r.'s of required line passing through (2,–1,–1) are 1, 2, 3.

$\therefore$ Cartesian equation of line is: $\frac{x-2}{1} = \frac{y-(-1)}{2} = \frac{z-(-1)}{3} \qquad \Rightarrow \frac{x-2}{1} = \frac{y+1}{2} = \frac{z+1}{3}$.

And, vector equation is: $\vec{r} = (2\hat{i}-\hat{j}-\hat{k}) + \lambda(\hat{i}+2\hat{j}+3\hat{k})$.

(OR) Let $L_1 : \vec{r} = (6\hat{i}+2\hat{j}+2\hat{k}) + \lambda(\hat{i}-2\hat{j}+2\hat{k})$ and $\vec{r} = (-4\hat{i}-\hat{k}) + \mu(3\hat{i}-2\hat{j}-2\hat{k})$.

$\therefore \vec{a}_1 = 6\hat{i}+2\hat{j}+2\hat{k}, \vec{b}_1 = \hat{i}-2\hat{j}+2\hat{k}, \vec{a}_2 = -4\hat{i}-\hat{k}, \vec{b}_2 = 3\hat{i}-2\hat{j}-2\hat{k}$

Now $\vec{b}_1 \times \vec{b}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -2 & 2 \\ 3 & -2 & -2 \end{vmatrix} = 8\hat{i}+8\hat{j}+4\hat{k}$ and, $\vec{a}_2 - \vec{a}_1 = -10\hat{i}-2\hat{j}-3\hat{k}$.

So, S.D. = $\left|\frac{(\vec{b}_1\times\vec{b}_2).(\vec{a}_2-\vec{a}_1)}{|\vec{b}_1\times\vec{b}_2|}\right|$

$= \left|\frac{(8\hat{i}+8\hat{j}+4\hat{k}).(-10\hat{i}-2\hat{j}-3\hat{k})}{\sqrt{8^2+8^2+4^2}}\right| = \left|\frac{-80-16-12}{\sqrt{8^2+8^2+4^2}}\right|$

$\therefore$ S.D. = 9 units.

Q15. Out of a group of 30 honest people, 20 always speak the truth. Two persons are selected at random from the group. Find the probability distribution of the number of selected persons who speak the truth. Also find the mean of the distribution. What values are described in this question?

Sol. Total no. of honest people = 30, No. of people speaking truth always = 20

$\therefore$ number of people who doesn't speak truth = 30 – 20 = 10.

Let X denotes the number of persons speaking the truth. So X = 0, 1, 2.

X	0	1	2
P(X)	$\frac{^{10}C_2}{^{30}C_2} = \frac{9}{87}$	$\frac{^{10}C_1 \times {}^{20}C_1}{^{30}C_2} = \frac{40}{87}$	$\frac{^{20}C_2}{^{30}C_2} = \frac{38}{87}$

Mean $= \sum_{i=0}^{2} X_i P(X_i) = 0 \times \frac{9}{87} + 1 \times \frac{40}{87} + 2 \times \frac{38}{87} = \frac{116}{87}$.

Values described here are**: Honesty and Truth.**

Q16. If $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$ then, prove that $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$.

(OR) If $y = (\sin x)^x + \sin^{-1}\sqrt{x}$, then find $\frac{dy}{dx}$.

Sol. We have $x = 2\cos\theta - \cos 2\theta$ and $y = 2\sin\theta - \sin 2\theta$.

Differentiating both w.r.t. θ both the sides, we get:

$\frac{dx}{d\theta} = -2\sin\theta + 2\sin 2\theta$ and, $\frac{dy}{d\theta} = 2\cos\theta - 2\cos 2\theta$

$\therefore \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx} = \frac{2(\cos\theta - \cos 2\theta)}{2(-\sin\theta + \sin 2\theta)}$ $\Rightarrow \frac{dy}{dx} = \frac{-2\sin\frac{3\theta}{2}\sin\left(-\frac{\theta}{2}\right)}{2\cos\frac{3\theta}{2}\sin\frac{\theta}{2}}$

i.e., $\frac{dy}{dx} = \tan\left(\frac{3\theta}{2}\right)$. **Hence Proved.**

(OR) Let $y = u + v$, where $u = (\sin x)^x$ and v = $v = \sin^{-1}\sqrt{x}$

Differentiating both the sides, we get: $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ …(i)

Now, $u = (\sin x)^x = e^{\log(\sin x)^x} = e^{x\log(\sin x)}$

$\Rightarrow \frac{du}{dx} = \frac{d}{dx}\left(e^{x\log(\sin x)}\right)$ $\Rightarrow \frac{du}{dx} = e^{x\log(\sin x)}\frac{d}{dx}\left(x\log(\sin x)\right)$

$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[x.\frac{d}{dx}\left(\log(\sin x)\right) + \log(\sin x).\frac{d}{dx}(x)\right]$

$\Rightarrow \frac{du}{dx} = (\sin x)^x\left[x.\cot x + \log(\sin x)\right]$ …(ii)

And, $v = \sin^{-1}\sqrt{x}$ $\Rightarrow \frac{dv}{dx} = \frac{d}{dx}\left(\sin^{-1}\sqrt{x}\right)$

$\Rightarrow \frac{dv}{dx} = \frac{1}{\sqrt{1-(\sqrt{x})^2}} \times \frac{d}{dx}\left(\sqrt{x}\right)$ $\Rightarrow \frac{dv}{dx} = \frac{1}{2\sqrt{x}\sqrt{1-x}}$ …(iii)

By (i), (ii) and (iii), we get:

$\frac{dy}{dx} = (\sin x)^x\left[x.\cot x + \log(\sin x)\right] + \frac{1}{2\sqrt{x}\sqrt{1-x}}$.

Q17. Using properties of determinants, prove that :

$$\begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c).$$

Sol. LHS: Let $\Delta = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$

$\Rightarrow \; = \begin{vmatrix} 0 & a-b & a^3-b^3 \\ 0 & b-c & b^3-c^3 \\ 1 & c & c^3 \end{vmatrix}$ [Applying $R_1 \to R_1 - R_2$ and $R_2 \to R_2 - R_3$

$\Rightarrow \ = (a-b)(b-c)\begin{vmatrix} 0 & 1 & a^2+ab+b^2 \\ 0 & 1 & b^2+bc+c^2 \\ 1 & c & c^3 \end{vmatrix}$ [Taking (a – b) & (b – c) common from R_1 & R_2

$\Rightarrow \ = (a-b)(b-c)\left[0-0+1.((b^2+bc+c^2)-(a^2+ab+b^2))\right]$ [On expanding along C_1

$\Rightarrow \ = (a-b)(b-c)\left[b(c-a)+(c-a)(c+a)\right]$

$\therefore \ \Delta = (a-b)(b-c)(c-a)(a+b+c) = \text{RHS}$. **Hence Proved.**

Q18. Prove that : $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{77}{36}$.

(OR) Solve for x : $\tan^{-1}3x+\tan^{-1}2x=\frac{\pi}{4}$.

Sol. LHS: $\sin^{-1}\frac{8}{17}+\sin^{-1}\frac{3}{5}=\tan^{-1}\frac{8}{15}+\tan^{-1}\frac{3}{4}$

$$=\tan^{-1}\left(\frac{\frac{8}{15}+\frac{3}{4}}{1-\frac{8}{15}\times\frac{3}{4}}\right)$$ [Using $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$$=\tan^{-1}\left(\frac{\frac{8\times4+3\times15}{15\times4}}{\frac{15\times4-8\times3}{15\times4}}\right)=\tan^{-1}\left(\frac{77}{36}\right)=\text{RHS}.$$ **Hence Proved.**

(OR) We have: $\tan^{-1}3x+\tan^{-1}2x=\frac{\pi}{4}$ [Using $\tan^{-1}x+\tan^{-1}y=\tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\Rightarrow \tan^{-1}\left(\frac{3x+2x}{1-(3x)(2x)}\right)=\frac{\pi}{4}$ $\Rightarrow \tan\tan^{-1}\left(\frac{5x}{1-6x^2}\right)=\tan\left(\frac{\pi}{4}\right)$

$\Rightarrow \frac{5x}{1-6x^2}=1$ $\Rightarrow 6x^2+5x-1=0$

$\Rightarrow 6x^2+6x-x-1=0$ $\Rightarrow (6x-1)(x+1)=0$

$\Rightarrow x=\frac{1}{6},\ x=-1$ Since x = –1 does not satisfy the given equation, so we reject it.

$\therefore \ x=\frac{1}{6}$.

Q19. Let A = R – {2} and B = R – {1}. If $f: A \to B$ is a function defined by $f(x)=\frac{x-1}{x-2}$, show that f is one-one and onto. Hence find f^{-1}.

Sol. Given $f(x)=\frac{x-1}{x-2}$, $f: A\to B$ where A = R – {2}, B = R – {1}.

For One-One: Let $x_1, x_2 \in A$.

Now, $f(x_1)=f(x_2)\Rightarrow \frac{x_1-1}{x_1-2}=\frac{x_2-1}{x_2-2}\Rightarrow 2x_1-3=2x_2-3$ i.e., $x_1=x_2$

$\therefore f$ is one-one function.

For Onto: Let $y\in B$ such that $y=f(x)$ i.e., $y=\frac{x-1}{x-2}\Rightarrow xy-2y=x-1\Rightarrow x=\frac{2y-1}{y-1}$

Since $x\in A$ for all $y\in B$, so f is onto function.

Hence f is bijective function with $f^{-1}=\frac{2y-1}{y-1}$.

Q20. If $y=\log[x+\sqrt{x^2+a^2}]$, then prove that $(x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$.

Sol. We have $y=\log[x+\sqrt{x^2+a^2}]$

Diff. w.r.t. x both sides, $\frac{dy}{dx}=\frac{d}{dx}\left(\log[x+\sqrt{x^2+a^2}]\right)$

$\frac{dy}{dx}=\frac{1}{[x+\sqrt{x^2+a^2}]}\times\frac{d}{dx}[x+\sqrt{x^2+a^2}]$ $\Rightarrow\frac{dy}{dx}=\frac{1}{[x+\sqrt{x^2+a^2}]}\times\left[1+\frac{1}{2\sqrt{x^2+a^2}}\times\frac{d}{dx}(x^2+a^2)\right]$

$\Rightarrow\frac{dy}{dx}=\frac{1}{[x+\sqrt{x^2+a^2}]}\times\left[1+\frac{1}{2\sqrt{x^2+a^2}}\times(2x+0)\right]$

$\Rightarrow\frac{dy}{dx}=\frac{1}{[x+\sqrt{x^2+a^2}]}\left[\frac{x+\sqrt{x^2+a^2}}{\sqrt{x^2+a^2}}\right]$ $\Rightarrow\frac{dy}{dx}=\frac{1}{\sqrt{x^2+a^2}}$ $\Rightarrow\sqrt{x^2+a^2}\frac{dy}{dx}=1$

Again diff. w.r.t. x both sides, $\Rightarrow\sqrt{x^2+a^2}\frac{d}{dx}\left(\frac{dy}{dx}\right)+\frac{dy}{dx}\frac{d}{dx}(\sqrt{x^2+a^2})=\frac{d}{dx}(1)$

$\Rightarrow\sqrt{x^2+a^2}\left(\frac{d^2y}{dx^2}\right)+\frac{dy}{dx}\left(\frac{2x}{2\sqrt{x^2+a^2}}\right)=0$

$\therefore\ (x^2+a^2)\frac{d^2y}{dx^2}+x\frac{dy}{dx}=0$. **Hence Proved.**

Q21. Evaluate : $\int\frac{3x+5}{x^3-x^2-x+1}dx$.

Sol. Let $I=\int\frac{3x+5}{x^3-x^2-x+1}dx$ $\Rightarrow I=\int\frac{3x+5}{x^2(x-1)-(x-1)}dx$

$\Rightarrow I=\int\frac{3x+5}{(x^2-1)(x-1)}dx$ $\Rightarrow I=\int\frac{3x+5}{(x+1)(x-1)^2}dx$

Consider $\frac{3x+5}{(x+1)(x-1)^2}=\frac{A}{(x-1)}+\frac{B}{(x-1)^2}+\frac{C}{(x+1)}$

$\Rightarrow 3x+5=A(x^2-1)+B(x+1)+C(x-1)^2$ [On equating the coeffs. of like terms from both sides

We get: $A=-\frac{1}{2},B=4,C=\frac{1}{2}$.

$\therefore\ I=\int\left[-\frac{1}{2(x-1)}+\frac{4}{(x-1)^2}+\frac{1}{2(x+1)}\right]dx$ i.e., $\therefore\ I=-\frac{1}{2}\log|x-1|-\frac{4}{x-1}+\frac{1}{2}\log|x+1|+k$

Hence, $I=\frac{1}{2}\log\left|\frac{x+1}{x-1}\right|-\frac{4}{x-1}+k$.

Q22. Dot product of a vector with vectors $\hat{i}-\hat{j}+\hat{k}$, $2\hat{i}+\hat{j}-3\hat{k}$ and $\hat{i}+\hat{j}+\hat{k}$ are respectively 4, 0 and 2. Find the vector.

Sol. Let $\vec{a}=\hat{i}-\hat{j}+\hat{k}$, $\vec{b}=2\hat{i}+\hat{j}-3\hat{k}$ and $\vec{c}=\hat{i}+\hat{j}+\hat{k}$. Let the required vector be $\vec{p}=x\hat{i}+y\hat{j}+z\hat{k}$.

According to question, $\vec{a}.\vec{p}=4\Rightarrow(\hat{i}-\hat{j}+\hat{k}).(x\hat{i}+y\hat{j}+z\hat{k})=4$ i.e., $x-y+z=4$ …(i)

Also, $\vec{b}.\vec{p}=0\Rightarrow(2\hat{i}+\hat{j}-3\hat{k}).(x\hat{i}+y\hat{j}+z\hat{k})=0$ i.e., $2x+y-3z=0$ …(ii)

And, $\vec{c}.\vec{p}=2\Rightarrow(\hat{i}+\hat{j}+\hat{k}).(x\hat{i}+y\hat{j}+z\hat{k})=2$ i.e., $x+y+z=2$ …(iii)

Solving the eqs. (i), (ii) & (iii), we get: $x=2$, $y=-1$, $z=1$. $\therefore$ Required vector $\vec{p}=2\hat{i}-\hat{j}+\hat{k}$.

SECTION – C

[Question numbers 23 to 29 carry 6 marks each.]

Q23. Show that the lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ are coplanar. Also find the equation of the plane containing them.

(OR) Find the distance of the point (1, –2, 3) from the plane x – y + z = 5 measured parallel to the line $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}$.

Sol. Let $L_1 : \vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $L_2 : \vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$.

We have $\vec{a}_1 = \hat{i} + \hat{j} - \hat{k}, \vec{b}_1 = 3\hat{i} - \hat{j}$, $\vec{a}_2 = 4\hat{i} - \hat{k}, \vec{b}_2 = 2\hat{i} + 3\hat{k}$.

For coplanarity of two lines, we must have $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$.

Consider $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = (3\hat{i} - \hat{j}).(-3\hat{i} - 9\hat{j} + 2\hat{k}) = -9 + 9 = 0$.

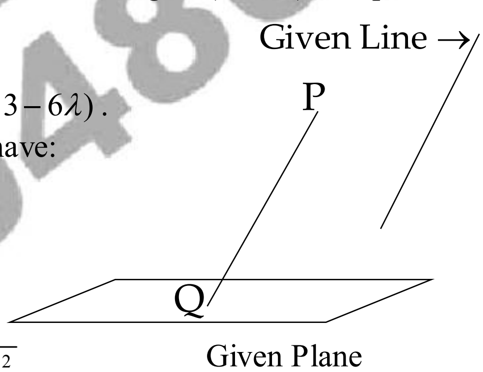
Hence lines L_1 and L_2 are coplanar.

Now for the required equation of plane, the normal to plane is $\vec{m} = \vec{b}_1 \times \vec{b}_2 = -3\hat{i} - 9\hat{j} + 2\hat{k}$.

$\therefore$ Equation of plane is: $(\vec{r} - (4\hat{i} - \hat{k})).(-3\hat{i} - 9\hat{j} + 2\hat{k}) = 0$ i.e., $\vec{r}.(3\hat{i} + 9\hat{j} - 2\hat{k}) = 14$.

(OR) Let P(1,–2, 3). The d.r.'s of $\frac{x-1}{2} = \frac{y-3}{3} = \frac{z+2}{-6}$ are 2, 3, –6.

Since parallel lines have proportionate d.r.'s so, equation of the line through P(1,–2,3) and parallel to the given line is: $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = \lambda$ ...(*i*).



Coordinates of any random point on line (i) is $Q(2\lambda + 1, 3\lambda - 2, 3 - 6\lambda)$.

If Q lies on the given equation of plane x – y + z = 5 then, we have:

$(2\lambda + 1) - (3\lambda - 2) + (3 - 6\lambda) = 5 \Rightarrow \lambda = \frac{1}{7}$.

So, coordinates of the point Q are $Q\left(\frac{9}{7}, -\frac{11}{7}, \frac{15}{7}\right)$.

$\therefore$ Required distance, $PQ = \sqrt{\left(\frac{9}{7} - 1\right)^2 + \left(-\frac{11}{7} + 2\right)^2 + \left(\frac{15}{7} - 3\right)^2} = 1$ unit.

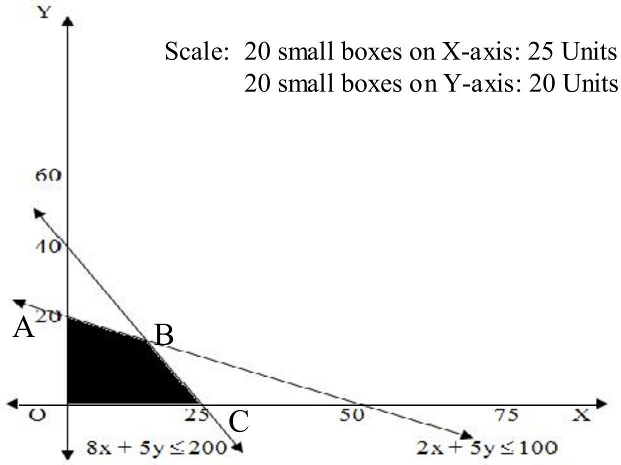
Q24. If a young man drives his scooter at 25kmph, he has to spend ₹2 per kilometer on petrol. If he drives the scooter at a speed of 40kmph, it produces more air pollution and increases his expenditure on petrol to ₹5 per kilometer. He has a maximum of ₹100 to spend on petrol and wishes to travel a maximum distance in one hour time with less pollution. Express this problem as an LPP and solve it graphically. What value do you find here?

Sol. Let the young man drives his scooter x km at a speed of 25kmph and y km at a speed of 40kmph.

To maximize: Z = (x + y) km

Subject to constraints: $x \ge 0, y \ge 0$, $2x + 5y \le 100 \Rightarrow \frac{x}{50} + \frac{y}{20} \le 1$ and, $\frac{x}{25} + \frac{y}{40} \le 1$.

Corner Points	Value of Z (in kms.)
O(0, 0)	0
A(0, 20)	20
B(50/3, 40/3)	30 ← Maximum
C(25, 0)	25



So, maximum value of Z is 30km when the young man drives $\frac{50}{3}$ km at 25km/hr and $\frac{40}{3}$ km at 40km/hr.

Values: Driving the motor vehicles at high speed rises the air pollution as well as the expenditures on fuel. So, one should avoid such practice.

Q25. In a group of 400 people, 160 are smokers and non-vegetarian, 100 are smokers and vegetarian and the remaining are non-smokers and vegetarian. The probabilities of getting a special chest diseases are 35%, 20% and 10% respectively. A person is chosen from the group at random and is found to be suffering from the disease. What is the probability that the selected person is a smoker and non-vegetarian? What value is reflected in this question?

Sol. Let E be the event of persons suffering from special chest disease.
Let E_1 be the event of those people who are smokers and non-vegetarian.
Let E_2 be the event of those people who are smokers and vegetarian.
Let E_3 be the event of those people who are non-smokers and vegetarian.

We have $P(E_1)=\frac{160}{400}, P(E_2)=\frac{100}{400}, P(E_3)=\frac{140}{400}$

And $P(E|E_1)=\frac{35}{100}, P(E|E_2)=\frac{20}{100}, P(E|E_3)=\frac{10}{100}$.

By using Bayes' theorem, we have: $P(E_1|E)=\frac{P(E|E_1)P(E_1)}{P(E|E_1)P(E_1)+P(E|E_2)P(E_2)+P(E|E_3)P(E_3)}$

$$\Rightarrow P(E_1|E)=\frac{\frac{35}{100}\times\frac{160}{400}}{\frac{35}{100}\times\frac{160}{400}+\frac{20}{100}\times\frac{100}{400}+\frac{10}{100}\times\frac{140}{400}}=\frac{35\times16}{35\times16+20\times10+10\times14}$$

$\therefore\ P(E_1|E)=\frac{28}{45}$.

Values: One should avoid smoking as it can be the reason of several dangerous diseases.

Q26. Two institutions decided to award their employees for the three values of resourcefulness, competence and determination in the form of prizes at the rate of ₹x, ₹y and ₹z respectively per person. The first institution decided to award respectively 4, 3 and 2 employees with a total prize money of ₹37000 and the second institution decided to award respectively 5, 3 and 4 employees with a total prize money of ₹47000. If all the three prizes per person together amount to ₹12000, then using matrix method find the value of x, y and z.
What values are described in this question?

Sol. By using the give information in the question, we have:

$$\begin{bmatrix}4&3&2\\5&3&4\\1&1&1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}37000\\47000\\12000\end{bmatrix}$$

i.e., $4x + 3y + 2z = 37000$, $5x + 3y + 4z = 47000$, $x + y + z = 12000$.

Let $A=\begin{bmatrix}4&3&2\\5&3&4\\1&1&1\end{bmatrix}, B=\begin{bmatrix}37000\\47000\\12000\end{bmatrix}, X=\begin{bmatrix}x\\y\\z\end{bmatrix}$

$\therefore AX=B\Rightarrow(A^{-1}A)X=A^{-1}B$ [Pre-multiplying by A^{-1} both sides

$\Rightarrow IX=A^{-1}B \qquad \Rightarrow X=A^{-1}B \quad \ldots$(i)

Now, $|A|=\begin{vmatrix}4&3&2\\5&3&4\\1&1&1\end{vmatrix}=4(3-4)-3(5-4)+2(5-3)=-3\neq0$ so, A^{-1} exists.

Consider C_{ij} as the cofactor of corresponding element $[a_{ij}]$ of matrix A.

$C_{11}=-1,\quad C_{12}=-1,\quad C_{13}=2,$
$C_{21}=-1,\quad C_{22}=2,\quad C_{23}=-1,$
$C_{31}=6,\quad C_{32}=-6,\quad C_{33}=-3$

$$\therefore adj\text{A} = \begin{bmatrix} -1 & -1 & 2 \\ -1 & 2 & -1 \\ 6 & -6 & -3 \end{bmatrix}^T \quad \Rightarrow adj\text{A} = \begin{bmatrix} -1 & -1 & 6 \\ -1 & 2 & -6 \\ 2 & -1 & -3 \end{bmatrix} \quad \text{So, } \text{A}^{-1} = \frac{1}{|\text{A}|} adj\text{A} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -6 \\ 1 & -2 & 6 \\ -2 & 1 & 3 \end{bmatrix}$$

By (i), we get: $\text{X} = \frac{1}{3}\begin{bmatrix} 1 & 1 & -6 \\ 1 & -2 & 6 \\ -2 & 1 & 3 \end{bmatrix}\begin{bmatrix} 37000 \\ 47000 \\ 12000 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 37000 + 47000 - 72000 \\ 37000 - 94000 + 72000 \\ -74000 + 47000 + 36000 \end{bmatrix} = \begin{bmatrix} 4000 \\ 5000 \\ 3000 \end{bmatrix}$$

By equality of matrices, we get: x = 4000, y = 5000, z = 3000.

The **values** described here are: **Resourcefulness, competence and determination.**

Q27. Find the equation of the tangent to the curve $y = \sqrt{3x - 2}$ which is parallel to the line $4x - 2y + 5 = 0$.

(OR) Prove that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$.

Sol. We have the equation of curve as $y = \sqrt{3x - 2}$ ...(i)

Differentiating w.r.t. x both sides, we get: $\frac{dy}{dx} = \frac{3}{2\sqrt{3x-2}}$

$\therefore$ Slope of tangent at P(x, y) is, $\left.\frac{dy}{dx}\right]_{\text{at } P(x,y)} = \frac{3}{2\sqrt{3x-2}} = m_T$.

Given line is $4x - 2y + 5 = 0 \Rightarrow y = 2x + \frac{5}{2}$

By comparing this line to y = mx + c, we get slope of line as $m_L = 2$.

Since parallel lines have same slope so, $m_L = 2 = m_T$

i.e., $\frac{3}{2\sqrt{3x-2}} = 2 \quad \Rightarrow \frac{3}{4} = \sqrt{3x-2} \quad \Rightarrow 2 + \frac{9}{16} = 3x$

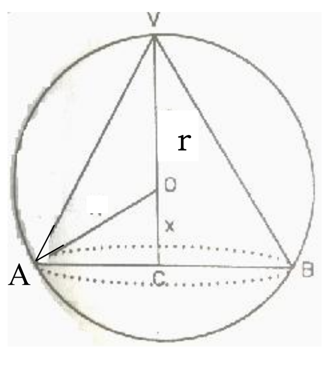
$\Rightarrow x = \frac{41}{48} \quad \therefore$ by (i) we get: $y = \sqrt{3\left(\frac{41}{48}\right) - 2} = \frac{3}{4} \quad \therefore$ Point of contact is $P\left(\frac{41}{48}, \frac{3}{4}\right)$.

So, equation of tangent is: $(y - y_1) = m_T(x - x_1)$

i.e., $y - \frac{3}{4} = 2\left(x - \frac{41}{48}\right) \quad \Rightarrow 48x - 24y = 23$.

(OR) Let VAB be a cone of maximum volume inscribed in a sphere of radius r.

Let OC = x. Then $AC = \sqrt{r^2 - x^2}$ = Radius of cone, VC = OC + VO = x + r = Height of cone.



Then volume of cone, $V = \frac{1}{3}\pi(AC)^2(VC)$

$\therefore V = \frac{1}{3}\pi(r^2 - x^2)(r + x)$

$\Rightarrow \frac{dV}{dx} = \frac{1}{3}\pi\left[r^2 - x^2 - 2x(r + x)\right] = \frac{1}{3}\pi\left[r^2 - 3x^2 - 2rx\right]$

And, $\frac{d^2V}{dx^2} = \frac{1}{3}\pi\left[-6x - 2r\right]$.

For points of local maxima & local minima, we have $\frac{dV}{dx} = 0$

i.e., $\frac{1}{3}\pi\left[r^2 - 3x^2 - 2rx\right] = 0 \Rightarrow (r-3x)(r+x) = 0 \quad \Rightarrow x = \frac{r}{3}, x = -r$

We shall reject $x = -r$. So, $\left.\frac{d^2V}{dx^2}\right]_{\text{at } x=\frac{r}{3}} = \frac{1}{3}\pi\left[-2r - 2r\right] = -\frac{4r\pi}{3} < 0$

So, V is maximum at $x = \frac{r}{3}$.

Now, height of cone VC = x + r $= \frac{r}{3} + r = \frac{4r}{3}$. **Hence Proved.**

Q28. Using integration, find the area of the region enclosed by the curves $y^2 = 4x$ and $y = x$.

Sol. We have $y^2 = 4x$ …(i) and, $y = x$ …(ii)

Solving (i) and (ii), we get (0, 0) and (4, 4) as their points of intersection.

So, required area $= \int_0^4 y_P\,dx - \int_0^4 y_L\,dx = \int_0^4 (2\sqrt{x} - x)\,dx$

$= \left(2 \times \frac{2}{3}\left[x^{3/2}\right]_0^4 - \frac{1}{2}\left[x^2\right]_0^4\right)$

$= \left(2 \times \frac{2}{3}\left[8 - 0\right] - \frac{1}{2}\left[16 - 0\right]\right)$

$\therefore$ Required Area $= \left(\frac{8}{3}\right)$ Sq.Units.

Q29. Find the particular solution of the following differential equation given that y = 0 when x = 1 : $(x^2 + xy)dy = (x^2 + y^2)dx$.

Sol. We have $(x^2 + xy)dy = (x^2 + y^2)dx$

$\Rightarrow \frac{dy}{dx} = \frac{(x^2 + y^2)}{(x^2 + xy)}$ …(i)

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

By (i): $v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{x^2 + x(vx)} \Rightarrow x\frac{dv}{dx} = \frac{1+v^2}{1+v}$

$\Rightarrow \int \frac{1+v}{1-v}dv = \int \frac{dx}{x}$

$\Rightarrow \int\left(-1 + \frac{2}{1-v}\right)dv = \int \frac{dx}{x}$

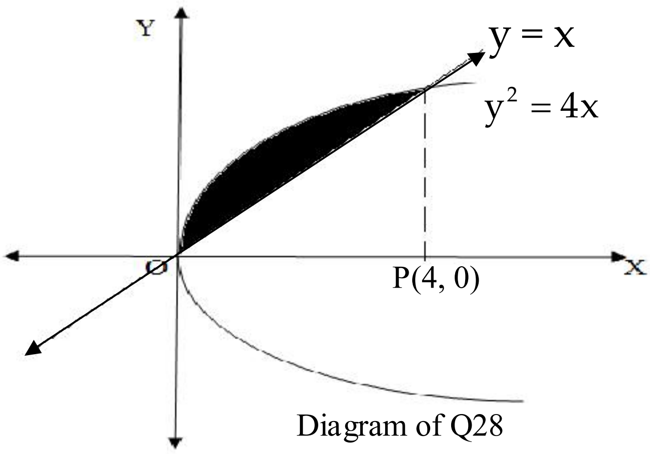


Diagram of Q28

$\Rightarrow -v - 2\log|1-v| = \log|x| + k \quad \Rightarrow -\frac{y}{x} - 2\log|x-y| + \log|x| = k$

As y (1) = 0 so, k = 0.

$\therefore$ the required solution of the differential equation is: $y = x\log\left|\frac{x}{(x-y)^2}\right|$.

Hii. Here is a short message I have to convey.

I've devoted myself for the service of Mathematics.. to help the students in need in all possible ways. It will be a thing of pleasure for me if my work/collection serves any purpose in your life.

Wish You All The Very Best! Lots of love and blessings!

– OP Gupta, INDIRA Award Winner [**M.:+91-9650350480, +91-9718240480, Mail at theopgupta@gmail.com**]