Complete Study Guide & Notes On
RELATIONS & FUNCTIONS

A Formulae Guide By OP Gupta (Indira Award Winner)

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RECAPITULATION & RELATIONS

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. TYPES OF INTERVALS

a) **Open interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying strictly between a and b is called an *open interval*. It is denoted by $]a,b[$ or (a,b) i.e., $\{x \in \mathrm{R} : a < x < b\}$.

b) **Closed interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes both a and b as well is known as a *closed interval*. It is denoted by $[a,b]$ i.e., $\{x \in \mathrm{R} : a \le x \le b\}$.

c) **Open Closed interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it excludes a and includes only b is known as an *open closed interval*. It is denoted by $]a,b]$ or $(a,b]$ i.e., $\{x \in \mathrm{R} : a < x \le b\}$.

d) **Closed Open interval:** If a and b be two real numbers such that $a < b$ then, the set of all the real numbers lying between a and b such that it includes only a and excludes b is known as a *closed open interval*. It is denoted by $[a,b[$ or $[a,b)$ i.e., $\{x \in \mathrm{R} : a \le x < b\}$.

RELATIONS

02. Defining the Relation: A relation R, from a non-empty set A to another non-empty set B is mathematically defined as an arbitrary subset of $\mathrm{A} \times \mathrm{B}$. Equivalently, any subset of $\mathrm{A} \times \mathrm{B}$ is a relation from A to B.

Thus, R is a relation from A to B $\Leftrightarrow \mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$

$$\Leftrightarrow \mathrm{R} \subseteq \{(a,b) : a \in \mathrm{A},\ b \in \mathrm{B}\}.$$

Illustrations:

a) Let $\mathrm{A} = \{1,2,4\}, \mathrm{B} = \{4,6\}$. Let $\mathrm{R} = \{(1,4),(1,6),(2,4),(2,6),(4,6)\}$. Here $\mathrm{R} \subseteq \mathrm{A} \times \mathrm{B}$ and therefore R is a relation from A to B.

b) Let $\mathrm{A} = \{1,2,3\}, \mathrm{B} = \{2,3,5,7\}$. Let $\mathrm{R} = \{(2,3),(3,5),(5,7)\}$. Here $\mathrm{R} \not\subset \mathrm{A} \times \mathrm{B}$ and therefore R is not a relation from A to B. Since $(5,7) \in \mathrm{R}$ but $(5,7) \notin \mathrm{A} \times \mathrm{B}$.

c) Let $\mathrm{A} = \{-1,1,2\}, \mathrm{B} = \{1,4,9,10\}$. Let $a \mathrm{R} b$ means $a^2 = b$ then, $\mathrm{R} = \{(-1,1),(1,1),(2,4)\}$.

Note the followings:

- *A relation from A to B is also called a relation from A into B.*
- $(a,b) \in R$ *is also written as* aRb *(read as* ***a is R related to b****).*
- *Let A and B be two non-empty finite sets having p and q elements respectively. Then* $n(A \times B) = n(A).n(B) = pq$. *Then total number of subsets of* $A \times B = 2^{pq}$. *Since each subset of* $A \times B$ *is a relation from A to B, therefore* ***total number of relations from A to B is given as*** 2^{pq}.

03. DOMAIN & RANGE OF A RELATION

a) **Domain of a relation:** Let R be a relation from A to B. The domain of relation R is the set of all those elements $a \in \mathrm{A}$ such that $(a,b) \in \mathrm{R}$ for some $b \in \mathrm{B}$. Domain of R is precisely written as $Dom.(R)$ symbolically.

Thus, $\mathrm{Dom.(R)} = \{a \in \mathrm{A} : (a,b) \in \mathrm{R} \text{ for some } b \in \mathrm{B}\}$.

That is, the domain of R is **the set of first component of all the ordered pairs which belong to R.**

b) **Range of a relation:** Let R be a relation from A to B. The range of relation R is the set of all those elements $b \in B$ such that $(a,b) \in R$ for some $a \in A$.

Thus, Range of $R = \{b \in B : (a,b) \in R \text{ for some } a \in A\}$.

That is, the range of R is the set of second components of all the ordered pairs which belong to R.

c) **Codomain of a relation:** Let R be a relation from A to B. Then B is called the codomain of the relation R. So we can observe that codomain of a relation R from A into B is the set B as a whole.

Illustrations:

a) *Let* $A = \{1,2,3,7\}, B = \{3,6\}$. *Let* aRb *means* $a < b$.

Then we have $R = \{(1,3),(1,6),(2,3),(2,6),(3,6)\}$.

Here $Dom.(R) = \{1,2,3\}$, *Range of* $R = \{3,6\}$, *Codomain of* $R = B = \{3,6\}$.

b) *Let* $A = \{1,2,3\}$, $B = \{2,4,6,8\}$. *Let* $R_1 = \{(1,2),(2,4),(3,6)\}$,

$R_2 = \{(2,4),(2,6),(3,8),(1,6)\}$. *Then both* R_1 *and* R_2 *are relations from A to B because* $R_1 \subseteq A \times B$, $R_2 \subseteq A \times B$. *Here* $Dom.(R_1) = \{1,2,3\}$, *Range of* $R_1 = \{2,4,6\}$;

$Dom.(R_2) = \{2,3,1\}$, *Range of* $R_2 = \{4,6,8\}$.

04. TYPES OF RELATIONS FROM ONE SET TO ANOTHER SET

a) **Empty relation:** A relation R from A to B is called an empty relation or a void relation from A to B if $R = \phi$.

For example, *let* $A = \{2,4,6\}, B = \{7,11\}$.

Let $R = \{(a,b) : a \in A, b \in B \text{ and } a - b \text{ is even}\}$. *Here R is an empty relation.*

b) **Universal relation:** A relation R from A to B is said to be the universal relation if $R = A \times B$.

For example, *let* $A = \{1,2\}, B = \{1,3\}$. *Let* $R = \{(1,1),(1,3),(2,1),(2,3)\}$. *Here* $R = A \times B$, *so relation R is a universal relation.*

05. RELATION ON A SET & ITS VARIOUS TYPES

A relation R from a non-empty set A into itself is called a relation on A. In other words if A is a non-empty set, then a subset of $A \times A = A^2$ is called a relation on A.

Illustrations:

Let $A = \{1,2,3\}$ *and* $R = \{(3,1),(3,2),(2,1)\}$. *Here R is relation on set A.*

<u>NOTE</u> If A be a finite set having *n* elements then, number of relations on set A is $2^{n \times n}$ ***i.e.,*** 2^{n^2}.

a) **Empty relation:** A relation R on a set A is said to be empty relation or a void relation if $R = \varphi$. In other words, a relation R in a set A is empty relation, if no element of A is related to any element of A, i.e., $R = \varphi \subset A \times A$.

For example, *let* $A = \{1,3\}$, $R = \{(a,b) : a \in A, b \in A \text{ and } a + b \text{ is odd}\}$. *Here R contains no element, therefore it is an empty relation on set A.*

b) **Universal relation:** A relation R on a set A is said to be the universal relation on A if $R = A \times A$ i.e., $R = A^2$. In other words, a relation R in a set A is universal relation, if each element of A is related to every element of A, i.e., $R = A \times A$.

For example, *let* $A = \{1,2\}$. *Let* $R = \{(1,1),(1,2),(2,1),(2,2)\}$. *Here* $R = A \times A$, *so relation R is universal relation on A.*

NOTE The void relation *i.e.*, ϕ and universal relation *i.e.*, $A \times A$ on A are respectively the *smallest* and *largest* relations defined on the set A. Also these are sometimes called *Trivial Relations*. And, any other relation is called a *non-trivial relation*.

❖ **The relations $R = \phi$ and $R = A \times A$ are two *extreme relations*.**

c) **Identity relation:** A relation R on a set A is said to be the identity relation on A if $R = \{(a,b) : a \in A, b \in A \text{ and } a = b\}$.

Thus identity relation $R = \{(a,a) : \forall\, a \in A\}$.

The identity relation on set A is also denoted by I_A.

For example, *let* $A = \{1,2,3,4\}$. *Then* $I_A = \{(1,1),(2,2),(3,3),(4,4)\}$. *But the relation given by* $R = \{(1,1),(2,2),(1,3),(4,4)\}$ *is not an identity relation because element 1 is related to elements 1 and 3.*

NOTE In an identity relation on A every element of A should be related to itself only.

d) **Reflexive relation:** A relation R on a set A is said to be reflexive if $a R a\ \forall a \in A$ i.e., $(a,a) \in R\ \forall a \in A$.

For example, *let* $A = \{1,2,3\}$, *and R_1, R_2, R_3 be the relations given as* $R_1 = \{(1,1),(2,2),(3,3)\}$, $R_2 = \{(1,1),(2,2),(3,3),(1,2),(2,1),(1,3)\}$ *and* $R_3 = \{(2,2),(2,3),(3,2),(1,1)\}$. *Here R_1 and R_2 are reflexive relations on A but R_3 is not reflexive as* $3 \in A$ *but* $(3,3) \notin R_3$.

NOTE The identity relation is always a reflexive relation but the opposite may or may not be true. As shown in the example above, R_1 is both identity as well as reflexive relation on A but R_2 is only reflexive relation on A.

e) **Symmetric relation:** A relation R defined on a set A is symmetric if $(a,b) \in R \Rightarrow (b,a) \in R\ \forall a,b \in A$ i.e., $a R b \Rightarrow b R a$ (i.e., whenever aRb then, bRa).

For example, *let* $A = \{1,2,3\}$, $R_1 = \{(1,2),(2,1)\}$, $R_2 = \{(1,2),(2,1),(1,3),(3,1)\}$, $R_3 = \{(2,3),(3,2),(2,2),(2,2)\}$ *i.e.* $R_3 = \{(2,3),(3,2),(2,2)\}$ *and* $R_4 = \{(2,3),(3,1),(1,3)\}$. *Here R_1, R_2 and R_3 are symmetric relations on A. But R_4 is not symmetric because* $(2,3) \in R_4$ *but* $(3,2) \notin R_4$.

f) **Transitive relation:** A relation R on a set A is transitive if $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$ i.e., $a R b$ and $b R c \Rightarrow a R c$.

For example, *let* $A = \{1,2,3\}$, $R_1 = \{(1,2),(2,3),(1,3),(3,2)\}$ *and* $R_2 = \{(1,3),(3,2),(1,2)\}$. *Here R_2 is transitive relation whereas R_1 is not transitive because* $(2,3) \in R_1$ *and* $(3,2) \in R_1$ *but* $(2,2) \notin R_1$.

g) **Equivalence relation:** Let A be a non-empty set, then a relation R on A is said to be an equivalence relation if

(i) R is reflexive i.e. $(a,a) \in R\ \forall a \in A$ i.e., aRa.

(ii) R is symmetric i.e. $(a,b) \in R \Rightarrow (b,a) \in R\ \forall a,b \in A$ i.e., $aRb \Rightarrow bRa$.

(iii) R is transitive i.e. $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R\ \forall a,b,c \in A$ i.e., aRb and $bRc \Rightarrow aRc$.

For example, *let* $A = \{1,2,3\}$, $R = \{(1,2),(1,1),(2,1),(2,2),(3,3)\}$. *Here R is reflexive, symmetric and transitive. So R is an equivalence relation on A.*

❖ ***Equivalence classes:*** Let A be an equivalence relation in a set A and let $a \in A$. Then, the set of all those elements of A which are related to a, is called equivalence class determined by a and it is denoted by $[a]$. Thus, $[a] = \{b \in A : (a, b) \in A\}$.

NOTE (i) Two equivalence classes are either disjoint or identical.
(ii) An equivalence relation R on a set A partitions the set into mutually disjoint equivalence classes.

An important property of an equivalence relation is that it divides the set into pair-wise disjoint subsets called **equivalence classes** whose collection is called **a partition of the set**. Note that the union of all equivalence classes gives the whole set.
e.g. Let R denotes the equivalence relation in the set Z of integers given by R = {(a,b) : 2 divides a – b}. Then the equivalence class [0] is [0] = $\{0, \pm 2, \pm 4, \pm 6, \ldots\}$

06. TABULAR REPRESENTATION OF A RELATION

In this form of representation of a relation R from set A to set B, elements of A and B are written in the first column and first row respectively. If $(a,b) \in R$ then we write '1' in the row containing a and column containing b and if $(a,b) \notin R$ then we write '0' in the same manner.

For example, *let* $A = \{1,2,3\}, B = \{2,5\}$ *and* $R = \{(1,2),(2,5),(3,2)\}$ *then,*

R	*2*	*5*
1	*1*	*0*
2	*0*	*1*
3	*1*	*0*

07. INVERSE RELATION

Let $R \subseteq A \times B$ be a relation from A to B. Then, the inverse relation of R, to be denoted by R^{-1}, is a relation from B to A defined by $R^{-1} = \{(b,a) : (a,b) \in R\}$.

Thus $(a,b) \in R \Leftrightarrow (b,a) \in R^{-1}\ \forall a \in A, b \in B$.

Clearly, Dom.(R^{-1}) = Range of R, Range of R^{-1} = Dom.(R).

Also, $(R^{-1})^{-1} = R$.

For example, *let* $A = \{1,2,4\}, B = \{3,0\}$ *and let* $R = \{(1,3),(4,0),(2,3)\}$ *be a relation from A to B then,* $R^{-1} = \{(3,1),(0,4),(3,2)\}$.

Summing up all the discussion given above, here is a recap of all these for quick grasp:

01.	**a)** A relation R from A to B is an empty relation or void relation iff $R = \varphi$.
	b) A relation R on a set A is an empty relation or void relation iff $R = \varphi$.
02.	**a)** A relation R from A to B is a universal relation iff $R = A \times B$.
	b) A relation R on a set A is a universal relation iff $R = A \times A$.
03.	A relation R on a set A is reflexive iff $aRa, \forall a \in A$.
04.	A relation R on set A is symmetric iff whenever aRb, then bRa for all $a, b \in A$.
05.	A relation R on a set A is transitive iff whenever aRb and bRc, then aRc.
06.	A relation R on A is identity relation iff $R = \{(a,a), \forall a \in A\}$ i.e., R contains only elements of the type $(a,a)\ \forall a \in A$ and it contains no other element.

07.	A relation R on a non-empty set A is an equivalence relation iff the following conditions are satisfied: i) **R is reflexive** i.e., for every $a \in \mathrm{A}$, $(a,a) \in \mathrm{R}$ i.e., $a\mathrm{R}a$. ii) **R is symmetric** i.e., for $a,b \in \mathrm{A}$, $a\mathrm{R}b \Rightarrow b\mathrm{R}a$ i.e., $(a,b) \in \mathrm{R} \Rightarrow (b,a) \in \mathrm{R}$. iii) **R is transitive** i.e., for all $a,b,c \in \mathrm{A}$ we have, $a\mathrm{R}b$ and $b\mathrm{R}c \Rightarrow a\mathrm{R}c$ i.e., $(a,b) \in \mathrm{R}$ and $(b,c) \in \mathrm{R} \Rightarrow (a,c) \in \mathrm{R}$.

FUNCTIONS & ITS VARIOUS TYPES

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. CONSTANT & TYPES OF VARIABLES

a) **Constant:** A constant is a symbol which retains the same value throughout a set of operations. So, a symbol which denotes a particular number is a constant. Constants are usually denoted by the symbols ***a, b, c, k, l, m,…*** etc.

b) **Variable:** It is a symbol which takes a number of values i.e., it can take any arbitrary values over the interval on which it has been defined. For example if x is a variable over R (set of real numbers) then we mean that x can denote any arbitrary real number. Variables are usually denoted by the symbols ***x, y, z, u, v,…*** etc.

c) **Independent variable:** That variable which can take an arbitrary value from a given set is termed as an independent variable.

d) **Dependent variable:** That variable whose value depends on the independent variable is called a dependent variable.

02. Defining A Function: Consider A and B be two non- empty sets then, a rule f which associates **each element of A with a unique element of B** is called a *function* or the *mapping from A to B* or *f maps A to B*. If f is a mapping from A to B then, we write $f : \mathrm{A} \to \mathrm{B}$ which is read as *'f is a mapping from A to B'* or *'f is a function from A to B'*.
If f associates $a \in \mathrm{A}$ to $b \in \mathrm{B}$, then we say that '***b* is the image of the element *a* under the function *f***' or '***b* is the *f* - image of *a***' or '**the value of *f* at *a***' and denote it by $f(a)$ and we write $b = f(a)$. The element a is called the **pre-image** or **inverse-image of *b***.

Thus for a function from A to B,
(i) A and B should be non-empty.
(ii) Each element of A should have image in B.
(iii) No element of A should have more than one image in B.
(iv) If A and B have respectively m and n number of elements then the **number of functions defined from A to B is n^m**.

03. Domain, Co-domain & Range of a function
The **set A is called the domain** of the function f and the **set B is called the co- domain**. The set of the images of all the elements of A under the function f is called the **range of the function *f*** and is denoted as $f(\mathrm{A})$.

Thus range of the function f is $f(\mathrm{A}) = \{f(x) : x \in \mathrm{A}\}$.
Clearly $f(\mathrm{A}) \subseteq \mathrm{B}$.

Note the followings:
(i) It is necessary that every f-image is in B; but there may be some elements in B which are not the f-images of any element of A i.e., whose pre-image under f is not in A.
(ii) Two or more elements of A may have same image in B.
(iii) $f : x \to y$ means that under the function f from A to B, an element x of A has image y in B.
(iv) Usually we denote the function f by writing $y = f(x)$ and read it as '***y* is a function of *x***'.

POINTS TO REMEMBER FOR FINDING THE DOMAIN & RANGE

Domain: If a function is expressed in the form $y = f(x)$, then domain of *f* means **set of all those real values of *x* for which *y* is real (i.e., *y* is well - defined)**.

❖ Remember the following points:

(i) *Negative number should not occur under the square root (even root) i.e., expression under the square root sign must be always* ≥ 0.

(ii) *Denominator should never be zero.*

(iii) *For* $log_b a$ *to be defined* $a > 0$, $b > 0$ *and* $b \neq 1$. *Also note that* $log_b 1$ *is equal to zero i.e.* 0.

Range: If a function is expressed in the form $y = f(x)$, then range of f means **set of all possible real values of *y* corresponding to every value of *x* in its domain**.

❖ Remember the following points:

(i) *Firstly find the domain of the given function.*

(ii) *If the domain does not contain an interval, then find the values of y putting these values of x from the domain. The set of all these values of y obtained will be the range.*

(iii) *If domain is the set of all real numbers R or set of all real numbers except a few points, then express x in terms of y and from this find the real values of y for which x is real and belongs to the domain.*

04. Function as a special type of relation: A relation *f* from a set A to another set B is said be a function (or mapping) from A to B if with every element (say *x*) of A, the relation *f* relates a unique element (say *y*) of B. This *y* is called *f*-image of *x*. Also *x* is called pre-image of *y* under *f*.

05. Difference between relation and function: A relation from a set A to another set B is any subset of $A \times B$; while a function *f* from A to B is a subset of $A \times B$ satisfying following conditions:

(i) For every $x \in A$, there exists $y \in B$ such that $(x, y) \in f$.

(ii) If $(x, y) \in f$ and $(x, z) \in f$ then, $y = z$.

Sl. No.	Function	Relation
01.	Each element of A must be related to some element of B.	There may be some element of A which are not related to any element of B.
02.	An element of A should not be related to more than one element of B.	An element of A may be related to more than one elements of B.

06. Real valued function of a real variable: If the domain and range of a function *f* are subsets of R (the set of real numbers), then *f* is said to be **a real valued function of a real variable** or **a real function**.

07. Some important real functions and their domain & range

FUNCTION	REPRESENTATION	DOMAIN	RANGE
a) Identity function	$I(x) = x \ \forall x \in R$	R	R
b) Modulus function or Absolute value function	$f(x) = \lvert x \rvert = \begin{cases} x & if \ x \geq 0 \\ -x & if \ x < 0 \end{cases}$	R	$[0, \infty)$
c) Greatest integer function or Integral function or Step function	$f(x) = [x]$ or $f(x) = \lfloor x \rfloor \ \forall x \in R$	R	Z
d) Smallest integer function	$f(x) = \lceil x \rceil \ \forall x \in R$	R	Z
e) Signum function	$f(x) = \begin{cases} \frac{\lvert x \rvert}{x} & if \ x \neq 0 \\ 0 & if \ x = 0 \end{cases}$ i.e., $f(x) = \begin{cases} 1, x > 0 \\ 0, x = 0 \\ -1, x < 0 \end{cases}$	R	$\{-1, 0, 1\}$

f) Exponential function	$f(x)=a^x \;\forall a \neq 1, a>0$	R	$(0,\infty)$
g) Logarithmic function	$f(x)=\log_a x \;\forall a \neq 1, a>0$ and $x>0$	$(0,\infty)$	R

08. TYPES OF FUNCTIONS

a) One-one function (Injective function or Injection): A function $f: \mathrm{A} \to \mathrm{B}$ is one- one function or injective function if distinct elements of A have distinct images in B.

Thus, $f: \mathrm{A} \to \mathrm{B}$ is one-one $\Leftrightarrow f(a)=f(b) \Rightarrow a=b \;\forall a,b \in \mathrm{A}$

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b) \;\forall a,b \in \mathrm{A}$.

❖ If A and B are two sets having m and n elements respectively such that $m \leq n$, then **total number of one-one functions** from set A to set B is ${}^n\mathrm{C}_m \times m!$ *i.e.*, ${}^n\mathrm{P}_m$.

❖ If n(A) = n then the number of injective functions defined from A onto itself is $n!$.

<u>ALGORITHM TO CHECK THE INJECTIVITY OF A FUNCTION</u>

STEP1- Take any two arbitrary elements *a, b* in the domain of *f*.

STEP2- Put $f(a)=f(b)$.

STEP3- Solve $f(a)=f(b)$. If it gives $a=b$ only, then *f* is a one-one function.

b) Onto function (Surjective function or Surjection): A function $f: \mathrm{A} \to \mathrm{B}$ is onto function or a surjective function if every element of B is the *f*-image of some element of A. That implies $f(\mathrm{A})=\mathrm{B}$ or range of *f* is the co-domain of *f*.

Thus, $f: \mathrm{A} \to \mathrm{B}$ is onto $\Leftrightarrow f(\mathrm{A})=\mathrm{B}$ *i.e.*, range of f = co-domain of f.

<u>ALGORITHM TO CHECK THE SURJECTIVITY OF A FUNCTION</u>

STEP1- Take an element $b \in \mathrm{B}$.

STEP2- Put $f(x)=b$.

STEP3- Solve the equation $f(x)=b$ for *x* and obtain *x* in terms of *b*. Let $x=g(b)$.

STEP4- If for all values of $b \in \mathrm{B}$, the values of *x* obtained from $x=g(b)$ are in A, then *f* is onto. If there are some $b \in \mathrm{B}$ for which values of *x,* given by $x=g(b)$, is not in A. Then *f* is not onto.

c) One-one onto function (Bijective function or Bijection): A function $f: \mathrm{A} \to \mathrm{B}$ is said to be one-one onto or bijective if it is both one-one and onto i.e., if the distinct elements of A have distinct images in B and each element of B is the image of some element of A.

❖ Also note that a **bijective function is also called a one-to-one function or one-to-one correspondence.**

❖ If $f: \mathrm{A} \to \mathrm{B}$ is a function such that,

i) f is one-one $\Rightarrow n(\mathrm{A}) \leq n(\mathrm{B})$.

ii) f is onto $\Rightarrow n(B) \leq n(\mathrm{A})$.

iii) f is one-one onto $\Rightarrow n(\mathrm{A}) = n(\mathrm{B})$.

❖ For an ordinary finite set A, a one-one function $f: \mathrm{A} \to \mathrm{A}$ is necessarily onto and an onto function $f: \mathrm{A} \to \mathrm{A}$ is necessarily one-one for every finite set A.

d) Identity Function: The function $I_A: \mathrm{A} \to \mathrm{A}$; $I_\mathrm{A}(x)=x \;\forall x \in \mathrm{A}$ is called an identity function on A.

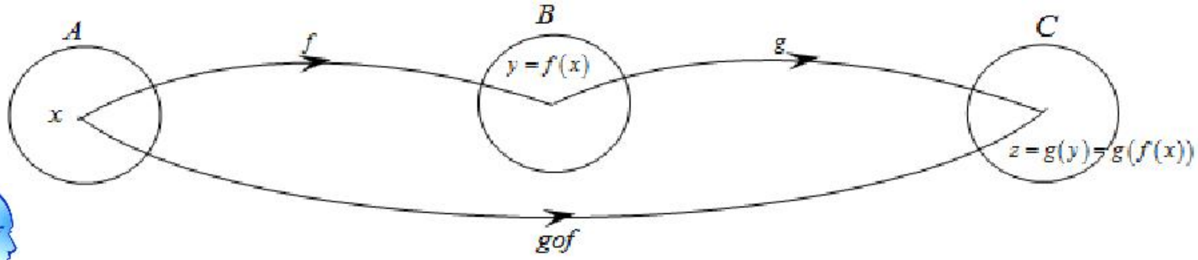
<u>NOTE</u> Domain$\left(I_\mathrm{A}\right)$ = A and Range$\left(I_\mathrm{A}\right)$ = A.

e) Equal Functions: Two function *f* and *g* having the same domain *D* are said to be equal if $f(x)=g(x)$ for all $x \in D$.

09. COMPOSITION OF FUNCTIONS

Let $f: \text{A} \to \text{B}$ and $g: \text{B} \to \text{C}$ be any two functions. Then f maps an element $x \in \text{A}$ to an element $f(x) = y \in \text{B}$; and this y is mapped by g to an element $z \in \text{C}$. Thus, $z = g(y) = g\left(f(x)\right)$. Therefore, we have a rule which associates with each $x \in \text{A}$, a unique element $z = g\left(f(x)\right)$ of C. This rule is therefore a mapping from A to C. We denote this mapping by gof (read as ***g composition f***) and call it the **'composite mapping of *f* and *g*'**.

Definition: *If $f: A \to B$ and $g: B \to C$ be any two mappings (functions), then the composite mapping gof of f and g is defined by $gof: A \to C$ such that $(gof)(x) = g\left(f(x)\right)$ for all $x \in A$.*



- The composite of two functions is also called the **resultant of two functions** or **the function of a function**.
- Note that for the composite function *gof* to exist, it is essential that range of *f* must be a subset of the domain of *g*.
- Observe that the order of events occur from the right to left i.e., *gof* reads composite of *f* and *g* and it means that we have to first apply *f* and then, follow it up with *g*.
- $Dom.(gof) = Dom.(f)$ and $Co\text{-}dom.(gof) = Co\text{-}dom.(g)$.
- Remember that gof is well defined only if $Co\text{-}dom.(f) = Dom.(g)$.
- If gof is defined then, it is not necessary that fog is also defined.
- If gof is one-one then f is also one-one function. Similarly, if gof is onto then g is onto function.

10. INVERSE OF A FUNCTION

Let $f: \text{A} \to \text{B}$ be a bijection. Then a function $g: \text{B} \to \text{A}$ which associates each element $y \in \text{B}$ to a unique element $x \in \text{A}$ such that $f(x) = y$ is called the inverse of *f* i.e., $f(x) = y \Leftrightarrow g(y) = x$.

The inverse of *f* is generally denoted by f^{-1}.

Thus, if $f: \text{A} \to \text{B}$ is a bijection, then a function $f^{-1}: \text{B} \to \text{A}$ is such that $f(x) = y \Leftrightarrow f^{-1}(y) = x$.

ALGORITHM TO FIND THE INVERSE OF A FUNCTION

STEP1- Put $f(x) = y$ where $y \in \text{B}$ and $x \in \text{A}$.

STEP2- Solve $f(x) = y$ to obtain x in terms of y.

STEP3- Replace x by $f^{-1}(y)$ in the relation obtained in STEP2.

STEP4- In order to get the required inverse of f i.e. $f^{-1}(x)$, replace y by x in the expression obtained in STEP3 i.e. in the expression $f^{-1}(y)$.

BINARY OPERATIONS

IMPORTANT TERMS, DEFINITIONS & RESULTS

01. Definition & Basic Understanding

An operation is a process which produces a new element from two given elements; e.g., addition, subtraction, multiplication and division of numbers. If the new element belongs to the same set to which the two given elements belong, the operation is called a binary operation.

DEFINITION: A binary operation (or binary composition) $\ast$ on a non-empty set A is a mapping which associates with each ordered pair $(a,b) \in A \times A$ a unique element of A. This unique element is denoted by $a \ast b$.

Thus, a binary operation $\ast$ on A is a mapping $\ast : A \times A \to A$ defined by $\ast(a,b) = a \ast b$.

Clearly $a \in A,\ b \in A \Rightarrow a \ast b \in A$.

NOTE If set A has m elements, then number of binary operations on A is $m^{m \times m}$ i.e., m^{m^2}.

Also if set A has m elements, then number of commutative binary operations on A is $m^{m \times (m-1)/2}$.

02. Terms related to binary operations

- A ***binary operation*** $\ast$ on a set A is a function $\ast$ from $A \times A$ to A.
- An operation $\ast$ on A is ***commutative*** if $a \ast b = b \ast a \quad \forall a,b \in A$.
- An operation $\ast$ on A is ***associative*** if $(a \ast b) \ast c = a \ast (b \ast c) \ \forall a,b,c \in A$.
- An operation $\ast$ on A is ***distributive*** if $a \ast (b \otimes c) = (a \ast b) \otimes (a \ast c) \ \forall a,b,c \in A$. Also an operation $\ast$ on A is ***distributive*** if $(b \otimes c) \ast a = (b \ast a) \otimes (c \ast a) \ \forall a,b,c \in A$.
- An ***identity element*** of a binary operation $\ast$ on A is an element $e \in A$ such that $e \ast a = a \ast e = a \ \ \forall a \in A$. Identity element for a binary operation, if it exists is **unique**.

03. Inverse of an element

Let $\ast$ be a binary operation on a non-empty set A and e be the identity element for the binary operation $\ast$. An element $b \in A$ is called the inverse element or simply inverse of a for binary operation $\ast$ if $a \ast b = b \ast a = e$.

Thus, an element $a \in A$ is invertible if and only if its inverse exists.